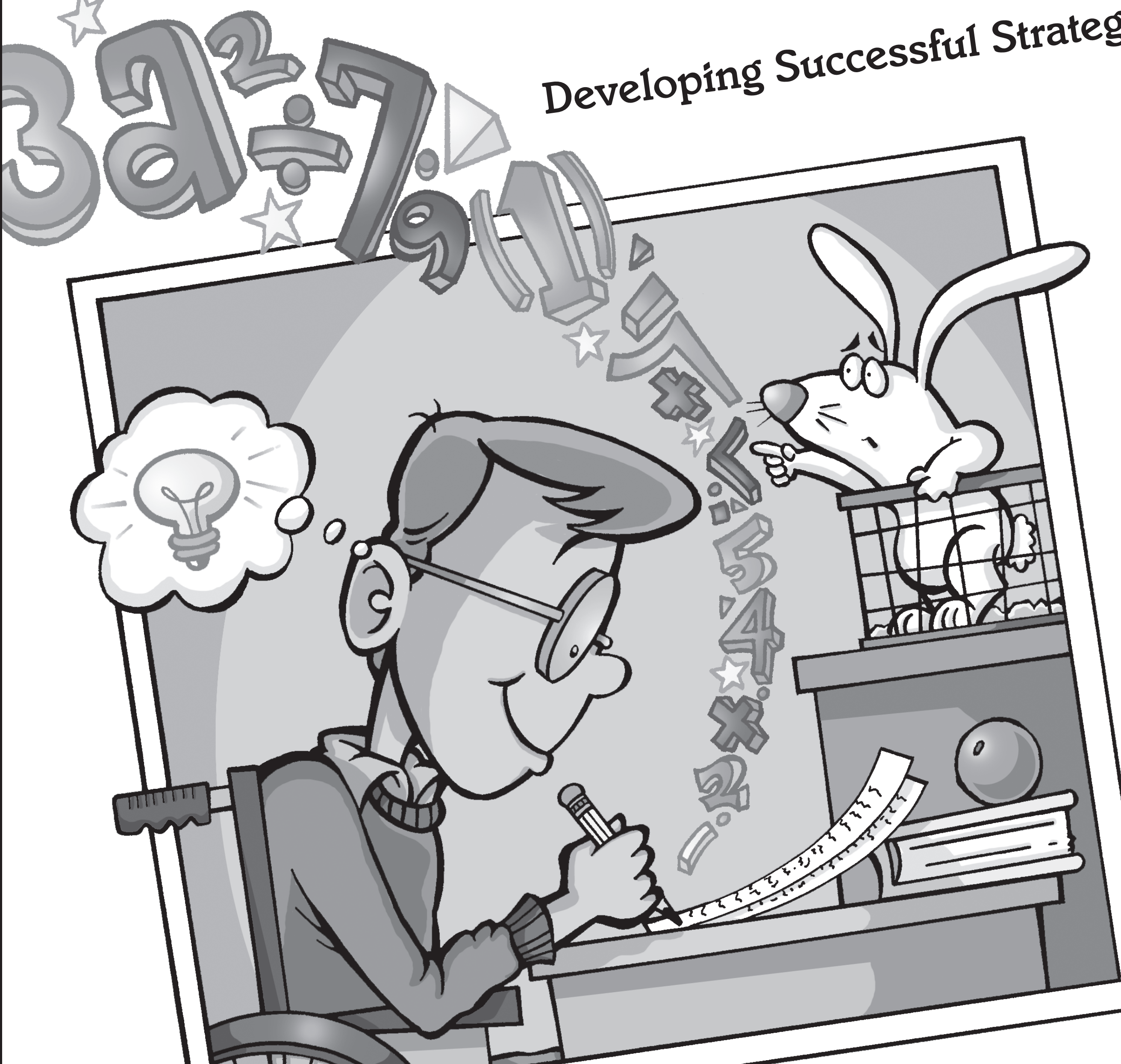
Grades 5–6

Math PROBLEM-SOLVING Skills

Developing Successful Strategies



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FOREWORD

The ***Math Problem-Solving Skills*** series has been developed to provide a rich resource for teachers of students from the elementary grades through middle school. The series of problems, discussions of ways to understand what is being asked, and means of obtaining solutions presented in these books aim to improve the problem-solving performance and persistence of all students. The authors believe it is critical that students and teachers engage with a few complex problems over an extended period rather than spend a short time on many straightforward problems or exercises. In particular, it is essential to allow students time to review and discuss what is required in the problem-solving process before moving to another and different problem. This series includes ideas for extending problems and solution strategies to help teachers implement this vital aspect of mathematics in their classrooms. The problems have been constructed and selected over many years of experience with students at all levels of mathematical talent and persistence, as well as in discussions with teachers in classrooms and professional learning and university settings.

Problem solving does not come easily to most people, so learners need many experiences engaging with problems if they are to develop this crucial ability. As they grapple with problem meaning and find solutions, students will learn a great deal about mathematics and mathematical reasoning. This leads to a focus on organizing what needs to be done rather than simply looking to apply one or more strategies.

Student and Teacher Pages

The student pages present problems chosen with a particular problem-solving focus and draw on a range of mathematical understandings and processes. For each set of related problems, teacher notes and discussion are provided. Answers to the more straightforward problems and detailed solutions to the more complex problems ensure appropriate explanations and suggest ways in which problems can be extended.

At the top of each teacher page, a statement highlights the particular thinking that the problems will demand, together with an indication of the mathematics that might be needed, a list of materials that can be used in seeking a solution, and the NCTM standards addressed. Each book is organized so that when a problem requires complicated strategic thinking, two or three problems occur on one page (supported by a teacher page with detailed discussion) to encourage students to find a solution together with a range of means that can be followed. More often, problems are grouped as a series of three interrelated pages where the level of complexity gradually increases, while the associated teacher page examines one or two of the problems in depth and highlights how the other problems might be solved in a similar manner.

Each teacher page concludes with two further aspects critical to the successful teaching of problem solving. A section on likely difficulties points to reasoning and content inadequacies that experience has shown may well impede students' success. In this way, teachers can be on the lookout for difficulties and be prepared to guide students past these potential pitfalls. The final section suggests extensions to the problems that can build a rich array of experiences with particular solution methods.

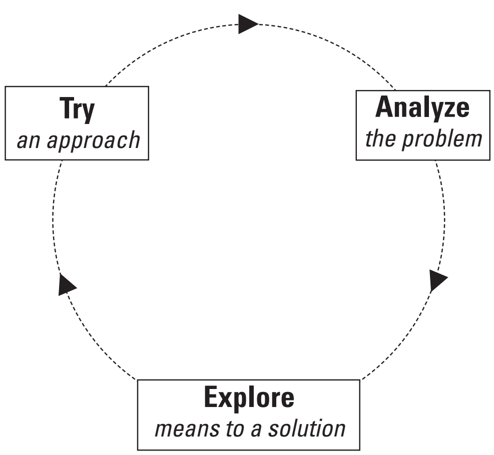
Mathematics and Language

The difficulty of the mathematics gradually increases over the series, largely in line with what is taught at the various grade levels, although problem solving both challenges at the point of the mathematics that is being learned and provides insights and motivation for what might be learned next.

FOREWORD

The language in which the problems are expressed is relatively straightforward, although this too increases in complexity across the series in terms of both the context in which the problems are set and the mathematical content that is required. It will always be a challenge for some students to "unpack" the meaning from a worded problem, particularly as the problems' context, information, and meanings expand. This ability is fundamental to the nature of mathematical problem solving and must be built up with time and experiences rather than diminished or left out of problem situations. It is suggested that students work in groups so that they can help one another tackle the ideas in complex problems through discussion, rather than simply leaping into the first ideas that come to mind (leaving the full extent of the problem unrealized).

An Approach to Solving Problems



The careful, gradual development of an ability to analyze problems for meaning, organize the information to make it meaningful, and make connections among problems to suggest a way forward to a solution is fundamental to the approach taken with this series. At first, materials are used explicitly to aid these meanings and connections; however, in time, they give way to diagrams, tables, and symbols as students' understanding of and experience with solving complex, engaging problems increases.

Not only is this model for the problem-solving process helpful in solving problems, but it also provides a basis for students to discuss their progress and solutions and determine whether or not they have fully answered a question. At the same time, it guides teachers' questions of students and provides a means of seeing underlying mathematical difficulties and ways in which problems can be adapted to suit particular needs and extensions. Above all, it provides a common framework for discussions between a teacher and group or among a whole class that focus on the problem-solving process rather than simply on the solution of particular problems. Indeed, as Alan Schoenfeld, in Steen, L. (Ed.), *Mathematics and Democracy* (2001), states so well, in problem solving:

Getting the answer is only the beginning rather than the end. … An ability to communicate thinking is equally important.

We wish all teachers and students who use these books success in fostering engagement with problem solving and building a greater capacity to come to terms with and solve mathematical problems at all levels.

George Booker and Denise Bond

Problem Solving and Mathematical Thinking

> *By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom. In everyday life and in the workplace, being a good problem solver can lead to great advantages.*
>
> **— NCTM Principles and Standards for School Mathematics (2000, p. 52)**

Problem solving lies at the heart of mathematics. New mathematical concepts and processes have always grown out of problem situations, and students' problem-solving capabilities develop from the very beginning of mathematics learning. A need to solve a problem can motivate students to acquire new ways of thinking as well as come to terms with concepts and processes that might not have been adequately learned when first introduced. Even those who can calculate efficiently and accurately are ill-prepared for a world where new and adaptable ways of thinking are essential if they are unable to identify which information or processes are needed.

On the other hand, students who can analyze the meaning of problems, explore means to a solution, and carry out a plan to solve mathematical problems have acquired deeper and more useful knowledge than simply being able to complete calculations, name shapes, use formulas to make measurements, or determine measures of chance and data. It is critical that mathematics teaching focuses on enabling all students to become both able and willing to engage with and solve mathematical problems.

Well-chosen problems encourage deeper exploration of mathematical ideas, build persistence, and highlight the need to understand thinking strategies, properties, and relationships. They also reveal the central role of *sense making* in mathematical thinking—not only to evaluate the need for assessing the reasonableness of an answer or solution, but also the need to consider the interrelationships among the information provided with a problem situation. This may take the form of number sense, allowing numbers to be represented in various ways and operations to be interconnected; through spatial sense that allows the visualization of a problem in both its parts and whole; to a sense of measurement across length, area, volume, and probability and data analysis.

Problem Solving

A problem is a task or situation for which there is no immediate or obvious solution, so that problem solving refers to the processes used when engaging with this task. When problem solving, students engage with situations for which a solution strategy is not immediately obvious, drawing on their understanding of concepts and processes they have already met, and will often develop new understandings and ways of thinking as they move toward a solution. It follows that a task that is a problem for one student may not be a problem for another and that a situation that is a problem at one level will only be an exercise or routine application of a known means to a solution at a later time.

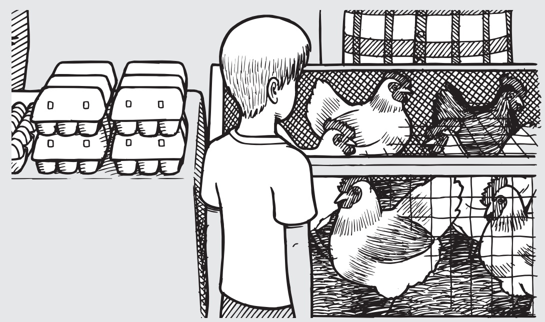
A large number of tourists visited Canyonlands National Park during 2007. There were twice as many visitors in 2007 as in 2003 and 6,530 more visitors in 2007 as in 2006. If there were 298,460 visitors in 2003, how many were there in 2006?

For a student in grades 3 or 4, sorting out the information to see how the number of visitors each year are linked is a considerable task. Multiplication and subtraction with large numbers are required. For student's in the upper elementary grades, an ability to see how the problem is structured and familiarity with computation could lead them to use a calculator, key in the numbers and operations in an appropriate order, and readily obtain the answer:

29,8460 × 2 – 6,530 = 590,390
590,390 tourists visited Canyonlands in 2006

As the world in which we live becomes ever more complex, the level of mathematical thinking and problem solving needed in life and in the workplace has increased considerably. To enable students to thrive in this changing world, attitudes and ways of knowing that help them to deal with new or unfamiliar tasks are now as essential as the procedures that have always been used to handle familiar operations readily and efficiently.

Such an attitude needs to develop from the beginning of mathematics learning as students form beliefs about meaning, the notion of taking control over the activities they engage with, and the results they obtain, and as they build an inclination to try different approaches. In other words, students need to see mathematics as a way of thinking rather than a means of providing answers to be judged right or wrong by a teacher, textbook, or some other external authority. They must be led to focus on ways of solving problems rather than on particular answers so that they understand the need to determine the meaning of a problem before beginning to work on a solution.



Lindsay sold 170 eggs at two different markets. He noticed that the number he sold at the second market was 10 fewer than half the number he sold at the first market. How many eggs did he sell at each market?

To solve this problem, it is not enough to simply use the numbers that are given. Rather, an analysis of the situation is needed first to see how the number sold at the second market relates to the number sold at the first market and the 170 eggs sold altogether. Putting the information into a diagram can help:

First market		Second market
Half number sold at first market	Half number sold at first market	10 less than half number sold at first market

The sum of the numbers in the three sections of the diagram is 170; half + half + (half – 10) = 170, so 3 × half = 180. Half the number sold at the first market is 60, so 120 eggs were sold at the first market and 50 eggs at the second. A diagram or the use of materials is needed first to interpret the situation and then to see how a solution can be obtained.

However, many students feel inadequate when they encounter problem-solving questions. They seem to have no idea of how to go about finding a solution and are unable to draw on the competencies they have learned in number, geometry, and measurement. Often these difficulties stem from underdeveloped concepts for the operations, spatial thinking, and measurement processes. They may also involve an underdeveloped capacity to read problems for meaning and a tendency to be led astray by the wording or numbers in a problem situation.

Their approach may then be simply to try a series of guesses or calculations rather than consider using a diagram or materials to come to terms with what the problem is asking and using a systematic approach to organize the information given and required in the task. It is this ability to analyze problems that is the key to problem solving, enabling decisions to be made about which mathematical processes to use, which information is needed, and which ways of proceeding are likely to lead to a solution.

Making Sense in Mathematics

Making sense of the mathematics being developed and used must be seen as the central concern of learning. This is important not only in coming to terms with problems and means to solutions but also in terms of bringing meaning, representation, and relationships among mathematical ideas to the forefront of thinking about and with mathematics. Making sensible interpretations of any results and determining which of several possibilities is more or equally likely is critical in problem solving.

Number sense, which involves being able to work with numbers comfortably and competently, is important in many aspects of problem solving: in making judgments, interpreting information, and communicating ways of thinking. It is based on a full understanding of numeration concepts such as zero, place value, and the renaming of numbers in equivalent forms, so that 207 can be seen as 20 tens and 7 ones as well as 2 hundreds and 7 ones (or that $\frac{5}{2}$, 2.5, and $2\frac{1}{2}$ are all names for the same fraction amount). Automatic, accurate access to basic facts also underpins number sense, not as an end in itself but rather as a means of combining with numeration concepts to allow manageable mental strategies and fluent processes for larger numbers. Well-understood concepts for the operations are essential in allowing relationships within a problem to be revealed and taken into account when framing a solution.

Number sense requires:

- understanding relationships among numbers
- appreciating the relative size of numbers
- a capacity to calculate and estimate mentally
- fluent processes for larger numbers and adaptive use of calculators
- an inclination to use understanding and facility with numeration and computation in flexible ways

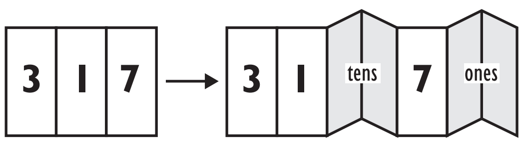
The following problem highlights the importance of these understandings.

There were 317 people at the New Year's Eve party on December 31. If each table could seat 5 couples, how many tables were needed?

Reading the problem carefully shows that each table seats five couples, or 10 people. At first glance, this problem might be solved using division; however, this would result in a decimal fraction, which is not useful in dealing with people seated at tables:

10)317 is 31.7

In contrast, a full understanding of numbers allows 317 to be renamed as 31 tens and 7 ones:



This provides for all the people at the party, and analysis of the number 317 shows that there have to be at least 32 tables for everyone to have a seat and allow partygoers to move around and sit with others during the evening. Understanding how to *rename* a number has provided a direct solution without any need for computation. It highlights how coming to terms with a problem and integrating this with number sense provides a means of solving the problem more directly and allows an appreciation of what the solution might mean.

Spatial sense is equally important, as information is frequently presented in visual formats that must be interpreted and processed, while the use of diagrams is often essential in developing conceptual understanding across all aspects of mathematics. Using diagrams, placing information in tables, or depicting a systematic way of dealing with the various possibilities in a problem assist in visualizing what is happening. It can be a very powerful tool in coming to terms with the information in a problem, and it provides insight into ways to proceed to a solution.

Spatial sense involves:

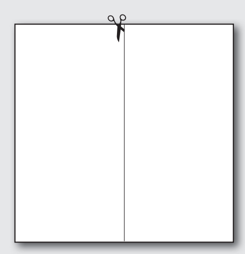
- a capacity to visualize shapes and their properties
- determining relationships among shapes and their properties
- linking two-dimensional and three-dimensional representations
- presenting and interpreting information in tables and lists
- an inclination to use diagrams and models to visualize problem situations and applications in flexible ways

The following problem shows how these under-standings can be used.

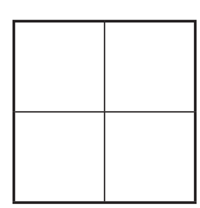
A small sheet of paper has been folded in half and then cut along the fold to make two rectangles.

The perimeter of each rectangle is 18 cm.

What was the perimeter of the original square sheet of paper?



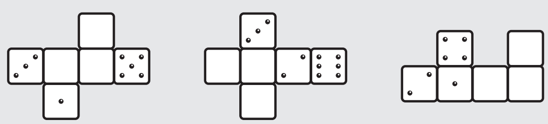
Reading the problem carefully and analyzing the diagram show that the length of the longer side of the rectangle is the same as the one side of the square, while the other side of the rectangle is half this length. Another way to obtain this insight is to make a square, fold it in half along the cutting line, and then fold it again. This shows that the large square is made up of four smaller squares:



Since each rectangle contains two small squares, the side of the rectangle, 18 cm, is the same as 6 sides of the smaller square, so the side of the small square is 3 cm. The perimeter of the large square is made of 6 of these small sides, so it is 24 cm.

Similar thinking is used with arrangements of two-dimensional and three-dimensional shapes and in visualizing how they can fit together or be taken apart.

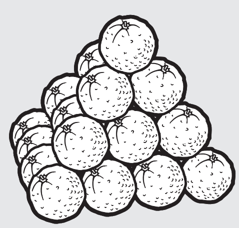
Many dice are made in the shape of a cube with arrangements of dots on each square face so that the sum of the dots on opposite faces is always 7. An arrangement of squares that can be folded to make a cube is called a net of a cube.



Which of these arrangements of squares forms a net for the dice?

Greengrocers often stack fruit as a pyramid.

How many oranges are in this stack?



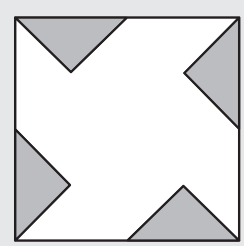
Measurement sense is dependent on both number sense and spatial sense, since attributes that are one-, two-, or three-dimensional are quantified to provide both exact and approximate measures and allow comparison. Many measurements use aspects of geometry (length, area, volume), while others use numbers on a scale (time, mass, temperature). Money can be viewed as a measure of value and uses numbers more directly, while practical activities such as map reading and determining angles require a sense of direction as well as gauging measurement. The coordination of the thinking for number and geometry, along with an understanding of how the metric system builds on place value, zero, and renaming, are critical in both building measurement understanding and using it to come to terms with and solve many practical problems and applications.

Measurement sense includes:

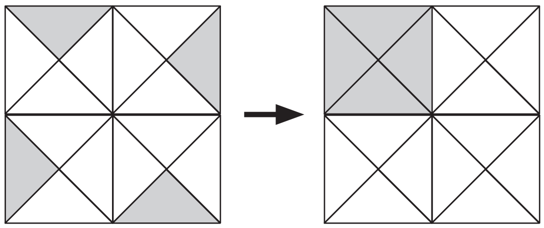
- understanding how numeration and computation underpin measurement
- extending relationships from number understanding to the metric system
- appreciating the relative size of measurements
- a capacity to use calculators and mental or written processes for exact and approximate calculations
- an inclination to use understanding and facility with measurements in flexible ways

The following problem shows how these understandings can be used.

A city square has an area of 160 m^2. Four small triangular garden beds are constructed from each corner to the midpoints of the sides of the square. What is the area of each garden bed?



Reading the problem carefully shows that there are four garden beds and each of them takes up the same proportions of the whole square. A quick look at the area of the square shows that there will not be an exact number of meters along one side. Some further thinking will be needed to determine the area of each garden bed.



If the midpoints of the four sides are connected across the square, four smaller squares are formed and each garden bed takes up $\frac{1}{4}$ of a small square. Four of the garden beds will have the same area as one small square. Since the area of the small square is $\frac{1}{4}$ the area of the large square, the area of one small square is 40 m^2 and the area of each triangular garden bed is 10 m^2.

An understanding of the problem situation given by a diagram has been integrated with spatial thinking and a capacity to calculate mentally with simple fractions to provide an appropriate solution. Both spatial sense and number sense have been used to understand the problem and suggest a means to a solution.

Data sense is an outgrowth of measurement sense and refers to an understanding of the way number sense, spatial sense, and a sense of measurement work together to deal with situations where patterns need to be discerned among data or when likely outcomes need to be analyzed. This can occur among frequencies in data or possibilities in chance.

Data sense involves:

- understanding how numeration and computation underpin the analysis of data
- appreciating the relative likelihood of outcomes
- a capacity to use calculators or mental and written processes for exact and approximate calculations
- presenting and interpreting data in tables and graphs
- an inclination to use understanding and facility with number combinations and arrangements in flexible ways

The following problem shows how these understandings can be used.

A bag has 5 blue marbles, 3 red marbles, and 4 yellow marbles. How many red marbles must be added to the bag so that the probability of drawing a red marble is $\frac{3}{4}$?

An understanding of probability and careful analysis of the situation are needed to come to terms with what the problem is asking. If the probability of drawing a red marble is $\frac{3}{4}$, then the probability of drawing a blue or yellow marble must be $\frac{1}{4}$. There are nine blue or yellow marbles, so there would have to be 36 marbles altogether to give the probability of $\frac{1}{4}$, and all the other marbles must be red. 27 of the marbles would have to be red, so another 24 red marbles must be added to the bag. A systematic consideration of the possible outcomes has made it possible to find a solution.

Patterning is another critical aspect of sense making in mathematics. Often a problem calls on discerning a pattern in the placement of materials, the numbers involved in the situation, or the possible arrangements of data or outcomes so as to determine a likely solution. Being able to see patterns is also very helpful in getting a solution more immediately or understanding whether or not a solution is complete.

A farmer has emus and alpacas in one paddock. When she counted, there were 38 heads and 100 legs. How many emus and how many alpacas are in the paddock?

There are 38 emus and alpacas. Emus have two legs. Alpacas have four legs.

Number of Alpacas	Number of Emus	Number of Legs
4	34	84 – too few
8	30	92 – too few
10	28	96 – too few
12	26	100

There are 12 alpacas and 26 emus.

As students gain more experience in solving problems, an ability to see patterns in what is occurring will also help them to obtain solutions more directly and see the relationship between a new problem and one that they have solved previously. It is this ability to relate problem types, even when the context appears to be quite different, that often distinguishes a good problem solver from one who is more hesitant.

Building a Problem-Solving Process

While the teaching of problem solving has often centered on the use of particular strategies that could apply to various classes of problems, many students are unable to access and use these strategies to solve problems outside of the teaching situations in which they were introduced. Rather than acquire a process for solving problems, they may attempt to memorize a set of procedures and view mathematics as a set of learned rules where success follows the use of the right procedure to the numbers given in the problem. Any use of strategies may be based on familiarity, personal preference, or recent exposure rather than through a consideration of the problem to be solved. A student may even feel it is sufficient to have only one strategy and that the strategy should work all of the time—and if it doesn't, then the problem can't be solved.

In contrast, observation of successful problem solvers shows that their success depends more on an analysis of the problem itself—what is being asked, what information might be used, what answer might be likely, and so on—so that a particular approach is used only after the intent of the problem is determined. Establishing the meaning of the problem before any plan is drawn up or work on a solution begins is critical. Students need to see that discussion about the problem's meaning and the ways of obtaining a solution must take precedence over a focus on the answer. Using collaborative groups when problem solving, rather than tasks assigned individually, is an approach that helps to develop this disposition.

Looking at a problem and working through what is needed to solve it will shed light on the problem-solving process.

Great-Grandma Jean left $93,000 in her will. She asked that it be divided so that each of her three great-grandchildren receive the same amount, their father (her grandson) twice as much as the three great-grandchildren together, and her daughter (the children's grandmother) $3,000 more than the father and great-grandchildren together. How much does each get?

Reading the problem carefully shows that Great-Grandma left her money to her daughter, grandson, and three great-grandchildren. She arranged her will so that her daughter was given $3,000 before the remaining $90,000 was distributed, so the amount given to the grandson is twice that given to her great-grandchildren, while the amount her daughter gets is equal to $3,000 plus the sum given to all the others. All of the information needed to solve the problem is available, and no further information is needed. The question at the end asks how much money each gets, but the problem is really about how the money is distributed among all the beneficiaries of the will.

The discussion of this problem serves to identify the key elements in the problem-solving process. To begin, it was necessary to *analyze* the problem to discover what must be considered. What a problem is really asking is rarely found in the problem statement. In this phase, it is necessary to look below the surface of the problem and come to terms with its structure. Reading the problem aloud, recalling previous difficult problems and other similar problems, selecting the important information, and discussing the problem's meaning are all essential.

The next step is to *explore* possible ways to solve the problem. If the analysis stage has been completed, then ways in which it might be solved will emerge. It is here that strategies and how they might be useful to solving a problem can arise. However, most problems can be solved in a variety of ways, using different approaches, and students must be encouraged to select a means that makes sense and appears achievable to them.

Possible ways that come to mind during the analysis are:

Materials – Counters could be used to represent each $1,000. Then work backwards through the problem from when $3,000 was kept aside for the daughter.

Try and adjust – Try an amount that the great-grandchildren might have received. Calculate the amounts and then adjust, if necessary, until the full $90,000 is allocated.

Backtrack using the numbers – The grandson received twice as much as the three great-grandchildren, while the daughter received as much as her grandson and great-grandchildren combined. The amount distributed to all the beneficiaries must be twice the amount given to the daughter.

Use a diagram to represent the information in the problem.

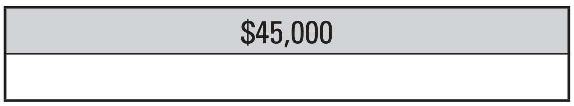
Think of a similar problem – For example, is it like a problem you have previously encountered and solved? If so, use similar reasoning to solve this problem.

Now one of the possible ways to a solution can be selected to *try*. Backtracking shows that $90,000 was double what was given to the daughter, so she must have received $45,000 in addition to the $3,000 already allocated. The other $45,000 was given to the grandson and three great-grandchildren, so the amount given to the grandson was double that given to the great-grandchildren. The grandson must have received $30,000, and the total given to the three great-grandchildren was half of this amount, or $15,000. Each great-grandchild must have been given $5,000.

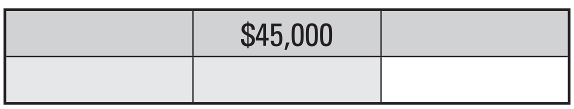
Materials could also have been used to work backwards. Ninety counters represent the $90,000 to be distributed, so 45 would be allocated to the daughter. The other 45 counters would have to be split so that the grandson gets twice as much as the great-grandchildren. Thirty counters must be given to the grandson and 15 to the great-grandchildren, so they would receive five counters each.

Another way to solve this problem is with a diagram. If we use a rectangle to represent the $90,000 left after the daughter got the additional $3,000, we can show this by shading how much was given to the others.

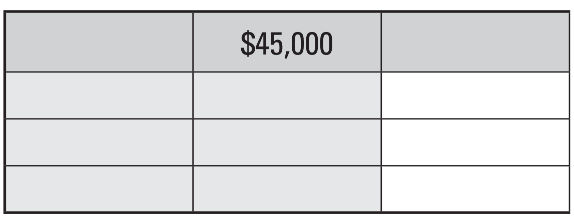
She received as much as the others combined, so she must have received half:



The father got twice as much as the great-grandchildren, so he received two-thirds of the remainder:



The great-grandchildren each received an equal share of the remaining $15,000:

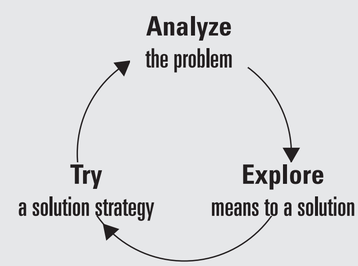


The nine equal parts together represent $45,000, so each great-grandchild received $5,000.

Having tried an idea, the answer(s) and solution must be analyzed in light of the problem in case another solution or answer is needed. It is essential to compare an answer to the original analysis of the problem to determine whether the solution obtained is reasonable. It will also raise the question of whether other answers exist and if there may be other solution strategies. In this way, the process is cyclic, and if the answer is unreasonable, the process will need to begin again.

In checking the solution, it is seen that if each great-grandchild received $5,000, then the grandson received twice as much as the three great-grandchildren ($30,000). The daughter received the same as the grandson and three great-grandchildren (or $45,000 and another $3,000). The total distributed is $93,000. Looking back at the problem, we see that this is correct and that the diagram has provided a means to the solution that has minimized and simplified the calculations.

Thinking about the various ways this problem could be solved highlights the key elements in the problem-solving process. When starting the process, it is necessary to ***analyze*** the problem to unfold its layers, discover its structure, and determine what the it is really asking. Next, all possible ways to solve the problem are ***explored*** before one or a combination of ways is selected to ***try***. Finally, once something has been tried, it is important to check the solution to see if it is reasonable. This process highlights the cyclic nature of problem solving and brings to the fore the importance of understanding the problem and its structure before proceeding. This process can be summarized as:



A Plan to Manage Problem Solving

This model provides students with a way of talking about the steps they engage in whenever they have a problem to solve. Discussing how they initially analyzed the problem, explored various ways that might provide a solution, and then tried one or more possible paths to obtain a solution—which they then analyzed for completeness and sense making—reinforces the very methods that will help them solve future problems. This process brings to the fore the importance of understanding the problem and its structure before proceeding.

Further, returning to an analysis of any answers and solution strategies highlights the importance of reflecting on what has been done. Taking time to reflect on any plans drawn up, processes followed, and strategies used brings out the significance of coming to terms with the nature of the problem, as well as the applicability of particular approaches to other problems.

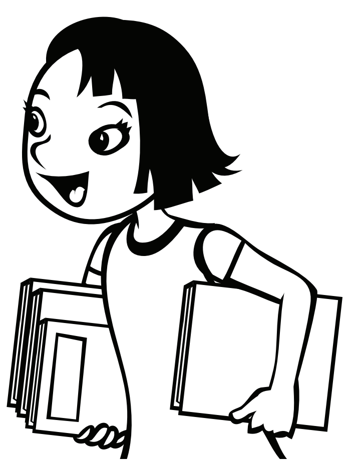
Thinking of how a related problem was solved is often the key to solving another problem at a later stage. It allows the thinking to be carried over to the new situation in a way that simply trying to think of the strategy used often fails to reveal. Analyzing problems in this way also highlights that a problem is not solved until the answer obtained can be justified. Learning to reflect on the *whole* process leads to the development of a deeper understanding of problem solving, and time must be allowed for reflection and discussion to fully build mathematical thinking.

Managing a Problem-Solving Program

Teaching problem solving differs from many other aspects of mathematics in that collaborative work can be more productive than individual work. Students who may be tempted to quickly give up when working on their own can be encouraged to see ways of proceeding when discussing a problem in a group, therefore building greater confidence in their capacity to solve problems and learning the value of persisting with a problem in order to tease out what is required. What is discussed with their peers is more likely to be recalled when other problems are met, while the observations made in the group increase the range of approaches that a student can access. Thus, time has to be allowed for discussion and exploration rather than insisting that students spend time on task, as for routine activities.

Correct answers that fully solve a problem are always important, but developing a capacity to use an effective problem-solving process must be the highest priority. Students who have an answer should be encouraged to discuss their solution with others who believe they have a solution, rather than tell their answer to another student or simply move on to another problem. In particular, explaining to others why they believe an answer is reasonable, as well as why it provides a solution, gets other students to focus on the entire problem-solving process rather than on just quickly getting an answer.

Expressing an answer in a sentence that relates to the question stated in the problem also encourages reflection on what was done and ensures that the focus is on solving the problem rather than providing an answer. These aspects of the teaching of problem solving should then be taken further, as particular groups discuss their solutions with the whole class and all students are able to participate in the discussion of the problem. In this way, problem solving as a way of thinking comes to the fore, rather than focusing on the answers as the main aim of their mathematical activities.



Questions must encourage students to explore possible means to a solution and try one or more of them, rather than point to a particular procedure. It can also help students to see how to progress in their thinking, rather than get stuck in a loop where the same steps are repeated over and over. While having too many questions that focus on the way to a solution may end up removing the problem-solving aspect from the question, having too few may cause students to become frustrated with the task and think that it is beyond them.

Students need to experience the challenge of problem solving and gain pleasure from working through the process that leads to a full solution. Taking time to listen to students as they try out their ideas, without comment and without directing them to a particular strategy, is also important. Listening provides a sense of how students' problem solving is developing, as assessing this aspect of mathematics can be difficult. After all, solving one problem will not necessarily lead to success on the next problem, nor will difficulty with a particular problem mean that the problems that follow will also be as challenging.

A teacher also may need to extend or adapt a given problem to ensure the problem-solving process is understood and can be used in other situations, instead of moving on to a different problem in another area of mathematics learning. This can help students to understand the importance of asking questions of a problem, as well as seeing how a way of thinking can be adapted to other related problems. Having students engage in this process of posing questions is another way of both assessing them and bringing them to terms with the overall process of solving problems.

Building a Problem-Solving Process

The cyclical model, *Analyze–Explore–Try*, provides a very helpful means of organizing and discussing possible solutions. However, care must be taken that it is not seen simply as a procedure to be memorized and then applied in a routine manner to every new problem. Rather, it must be carefully developed over a range of different problems, highlighting the components that are developed with each new problem.

Analyze

- As students read a problem, the need to first read for the *meaning* of the problem can be stressed. This may require reading more than once and can be helped by asking students to state in their own words what the problem is asking them to do.
- Further reading will be needed to sort out which information is needed and whether some is not needed or if other information must be gathered from the problem's context (for example, data presented within the illustration or table accompanying the problem) or whether the students' mathematical understandings must be used to find other relationships among the information. As the form of the problems becomes more complex, this thinking will be extended to incorporate further ways of dealing with the information; for example, measurement units, fractions, and larger numbers might need to be renamed to the same mathematical form.
- Thinking about any processes that might be needed and the order in which they are used, as well as the type of answer that could result, should also be developed in the context of new levels of problem structure.

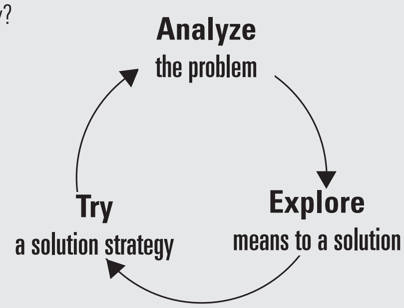
- Developing a capacity to see through the problem's expression—or context—to see similarities between new problems and others that might already have been met is a critical way of building expertise in coming to terms with and solving problems.

Expanding the Problem-Solving Process

A fuller model to manage problem solving can gradually emerge:

- Read carefully.
- What is the problem asking?
- What is the meaning of the information? Is it all needed? Is there too little? Too much?
- Which operations will be needed and in what order?
- What sort of answer is likely?
- Have I seen a problem like this before?

- Put the solution back into the problem.
- Does the answer make sense?
- Does it solve the problem?
- Is it the only answer?
- Could there be another way?



- Use materials or a model.
- Use a calculator.
- Use pencil and paper.
- Look for a pattern.

- Use a diagram or materials.
- Work backwards or backtrack.
- Put the information into a table.
- Try and adjust.

Explore

- When a problem is being explored, some problems will require the use of materials to think through the whole of the problem's context. Others will demand the use of diagrams to show what is needed. Still others will require a systematic analysis of the situation using a sequence of diagrams, or a list or table. As these ways of thinking about the problem are understood, they can be included in the cycle of steps.

Try

- Many students often try to guess a result. This can even be encouraged by talking about "guess and check" as a way of solving problems. Changing to "try and adjust" is more helpful in building a way of thinking and can lead to a very powerful method of finding solutions.
- When materials, a diagram, or a table has been used, another means to a solution is to look for a pattern in the results. When these have revealed what is needed to try for a solution, it may also be reasonable to use pencil and paper or a calculator.

Analyze

- The point in the cycle where an answer is assessed for reasonableness (for example, whether it provides a solution, is only one of several solutions, or there may be another way to solve the problem) also needs to be brought to the fore as different problems are met.

The Role of Calculators

When calculators are used, students devote less time to basic calculations, providing time that might be used to either explore a solution or find an answer to a problem. In this way, attention is shifted from computation, which the calculator can do, to thinking about the problem and its solution—work that the calculator cannot do. It also allows more realistic problems to be addressed in problem-solving sessions. In these situations, a calculator serves as a tool rather than a crutch, requiring students to think through the problem's solution in order to know how to use the calculator appropriately. It also underpins the need to make sense of the steps along the way and any answers that result, as keying incorrect numbers, operations, or order of operations quickly leads to results that are not appropriate.

Choosing, Adapting, and Extending Problems

When problems are selected, they need to be examined to see if students already have an understanding of the underlying mathematics required and that the problem's expression can be meaningfully read by the group of students who will be attempting the solution—though not necessarily by *all* students in the group. The problem itself should be neither too easy (so that it is just an exercise, repeating something readily done before), nor too difficult (thus beyond the capabilities of most or all in the group). A problem should engage the interest of the students and also be able to be solved in more than one way.

As a problem and its solution are reviewed, posing similar questions—where the numbers, shapes, or measurements are changed—focuses attention back on what was entailed in analyzing the problem and in exploring the means to a solution. Extending these processes to more complex situations shows how the particular approach can be extended to other situations and how patterns can be analyzed to obtain more general methods or results. It also highlights the importance of a systematic approach when conceiving and discussing a solution and can lead students to ask themselves further questions about the situation and to pose problems of their own as the significance of the problem's structure is uncovered.

Problem Structure and Expression

When analyzing a problem, it is also possible to discern critical aspects of the problem's form and relate this to an appropriate level of mathematics and problem expression when choosing or extending problems. A problem of first-level complexity uses simple mathematics and simple language. A second-level problem may have simple language and more difficult mathematics or more difficult language and simple mathematics, while a third-level problem has yet more difficult language and mathematics. Within a problem, the processes that must be used may be more or less obvious, the information that is required for a solution may be too much or too little, and strategic thinking may be needed in order to come to terms with what the problem is asking.

Level	processes obvious	processes less obvious	too much information	too little information	strategic thinking
increasing difficulty with problem's expression and mathematics required	simple expression, simple mathematics				
	more complex expression, simple mathematics simple expression, more complex mathematics				
	complex expression, complex mathematics				

The Varying Levels of Problem Structure and Expression

(i) The processes to be used are relatively obvious; these problems are comparatively straightforward and contain all the information necessary to find a solution.

(ii) The processes required are not immediately obvious; these problems contain all the information necessary to find a solution but demand further analysis to sort out what is wanted, and students may need to reverse what initially seemed to be required.

(iii) The problem contains more information than is needed for a solution, since these problems contain not only all the information needed to find a solution but also additional information in the form of times, numbers, shapes, or measurements.

(iv) Further information must be gathered and applied to the problem in order to obtain a solution. These problems do not contain all the information necessary to find a solution but do contain a means to obtain the required information. The problem's setting, the student's mathematical understanding, or the problem's wording must be searched for the additional material.

(v) Strategic thinking is required to analyze the question in order to determine a solution strategy. Deeper analysis, often aided by the use of diagrams or tables, is needed to come to terms with what the problem is asking so a means to a solution can be determined.

This analysis of the nature of problems can also serve as a means of evaluating the provision of problems within a mathematics program. In particular, it can lead to the development of a full range of problems, ensuring they are included across all problem forms, with the mathematics and expression suited to the level of the students.

Assessing Problem Solving

Assessment of problem solving requires careful and close observation of students working in a problem-solving setting. These observations can reveal the range of problem forms and the level of complexity in the expression and underlying mathematics that a student is able to confidently deal with. Further analysis of these observations can show to what extent the student is able to analyze the question, explore ways to a solution, select one or more methods to try, and then analyze any results obtained.

It is the combination of two fundamental aspects—the types of problem that can be solved and the manner in which solutions are carried out—that will give a measure of a student's developing problem-solving abilities rather than a one-off test in which some problems are solved and others are not.

Assessment informs:

What?
- *Problem form*
- *Problem expression*
- *Mathematics required*

How?
- *Analyze*
- *Explore*
- *Try*

Observations based on this analysis have led to a categorization of many of the possible difficulties that students experience with problem solving as a whole, rather than the misconceptions they may have about particular problems. These often involve inappropriate

Problem	***Likely Causes***
Student is unable to make any attempt at a solution.	• not interested • feels overwhelmed • cannot think of how to start to answer question • needs to reconsider complexity of steps and information
Student has no means of linking the situation to the implicit mathematical meaning.	• needs to create diagram or use materials • needs to consider separate parts of question and then bring parts together
Students uses an inappropriate operation.	• misled by word cues or numbers • has underdeveloped concepts • uses rote procedures rather than real understanding
Student is unable to translate a problem into a more familiar process.	• cannot see interactions between operations • lack of understanding means he/she is unable to reverse situations • data may need to be used in an order not evident in the problem statement or in an order contrary to that in which it is presented

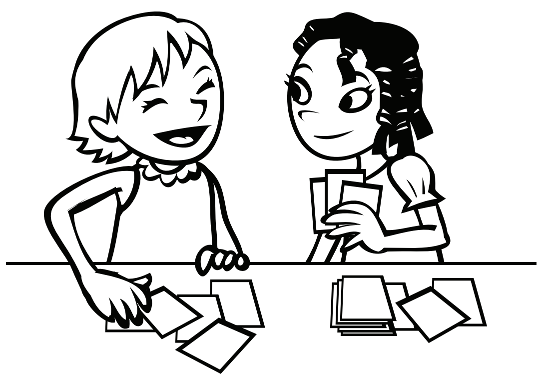
attempts at a solution based on little understanding of the problem.

A major cause of possible difficulties is the *lack of a well-developed plan of attack,* leading students to focus on the *surface level* of problems. In such cases, students:

- locate and manipulate numbers with little or no thought about their relevance to the problem
- try a succession of different operations if the first ones attempted do not yield a (likely) result
- focus on keywords for an indication of what might be done without considering their significance within the problem as a whole
- read problems quickly and cursorily to locate the numbers to be used
- use the first available word cue to suggest the operation that might be needed.

Other possible difficulties result from a focus on being quick, which leads to:

- no attempt to assess the reasonableness of an answer
- little perseverance if an answer is not obtained using the first approach tried
- not being able to access strategies to which they have been introduced



When the approaches to problem processing developed in this series are followed and the specific suggestions for solving particular problems or types of problems are discussed with students, these difficulties can be minimized, if not entirely avoided. Analyzing the problem before starting leads to an understanding of the problem's meanings. The cycle of steps within the model means that nothing is tried before the intent of the problem is clear and the means to a solution has been considered. Focusing on a problem's meaning and discussing what needs to be done builds perseverance. Making sense of the steps that must be followed and any answers that result are central to the problem-solving process. These difficulties are unlikely to occur among those who have built up an understanding of this way of thinking.

A Final Comment

If an approach to problem solving can be built up using the ideas developed here and the problems in the investigations on the pages that follow, students will develop a way of thinking about and with mathematics that will allow them to readily solve problems and generalize from what they already know to understand new mathematical ideas. They will engage with these emerging mathematical conceptions from their very beginnings, be prepared to debate and discuss their own ideas, and develop attitudes that will allow them to tackle new problems and topics. Mathematics can then be a subject that is readily engaged with and can become one in which the student feels in control, instead of one in which many rules devoid of meaning have to be memorized and applied at the right time. This early enthusiasm for learning and the ability to think mathematically will lead to a search for meaning in new situations and processes that will allow mathematical ideas to be used across a range of applications in school and everyday life.

A NOTE ON CALCULATOR USE

Many of the problems in this series require the use of a number of consecutive calculations, often requiring adding, subtracting, multiplying, or dividing the same amount to complete entries in a table or see a pattern. This demands (or will build) a certain amount of sophisticated use of the memory and constant functions of a simple calculator.

1. To add a number such as 9 repeatedly, it is sufficient on most calculators to enter an initial number (e.g., 30) and then press + 9 = = = = to add 9 over and over.
   - *30, 39, 48, 57, 66, …*
   - *To add 9 to a range of numbers, enter the first number (e.g., 30) and then press + 9 =. 30 + 9 = 39; 7 = gives 16; 3 = gives 12; 21 = gives 30; …*
   - *These are the answers when 9 is added to each number.*

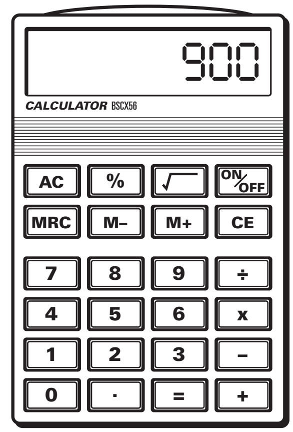
2. To subtract a number such as 5 repeatedly, it is sufficient on most calculators to enter an initial number (e.g., 92) and then press – 5 = = = = to subtract 5 over and over.
   - *92, 87, 82, 77, 72, …*
   - *To subtract 5 from a range of numbers, enter the first number (e.g., 92) and then press – 5 =. 92 – 5 = 87; 68 = gives 63; 43 = gives 38; 72 = gives 67; …*
   - *These are the answers when 5 is subtracted from each number.*

3. To multiply a number such as 10 repeatedly, most calculators now reverse the order in which the numbers are entered. Enter 10 × and then press an initial number (e.g., 15) = = = = to multiply by 10 over and over.
   - *150, 1,500, 15,000, 150,000, …*
   - *These are the answers when the given number is multiplied by 10.*
   - *This also allows squaring of numbers: 4 × = gives 16 or 4^2.*
   - *Continuing to press = gives more powers:*
   - *4 × = = gives 4^3 or 64; 4 × = = = gives 4^4; 4 × = = = = gives 4^5, and so on.*
   - *To multiply a range of numbers by 10, enter 10 × and then the first number (e.g., 90) and =.*
   - *10 × 90 = 900; 45 = gives 450; 21 = gives 210; 162 = gives 1,620; …*
   - *These are the answers when each number is multiplied by 10.*

4. To divide by a number such as 4 repeatedly, enter a number (e.g., 128).
   - *Then press ÷ 4 = = = = to divide each result by 4.*
   - *32, 8, 2, 0.5, …*
   - *These are the answers when the given number is divided by 4.*
   - *To divide a range of numbers by 4, enter the first number (e.g., 128) and ÷ 4 =. 128 ÷ 4 = 32; 64 = gives 16; 32 = gives 8; 12 = gives 3; …*
   - *These are the answers when each number is divided by 4.*

5. Using the memory keys M+, M–, and MR will also simplify calculations. A result can be calculated and added to memory (M+). Then a second result can be calculated and added to (M+) or subtracted from (M–) the result in the memory. Pressing MR will display the result. Often this will need to be performed for several examples as they are entered onto a table or patterns are explored directly.
   Clearing the memory after each completed calculation is essential!

   A number of calculations may also need to be made before addition, subtraction, multiplication, or division with a given number. That number can be placed in memory and used each time without having to rekey it.



6. The % key can be used to find percentage increases and decreases directly.
   - *To increase or decrease a number by a certain percent (e.g., 20%), simply key in the number and press = 20% or – 20% to get the answer:*
   - *80 + 20% gives 96 (not 100). 20% of 80 is 16; 80 + 16 is 96.*
   - *90 – 20% gives 72 (not 70). 20% of 90 is 18; 90 – 18 is 72.*

7. While the square root key can be used directly, finding other roots is best done by a "try and adjust" approach using the multiplication constant described above (in point 3).

MEETING THE NCTM STANDARDS

Numbers and Operations	
1.1 Understand numbers, ways of representing numbers, relationships among numbers, and number systems	pp. 24, 28, 32, 36, 48, 50, 52, 56, 60, 62, 64, 68, 72, 82, 86, 94, 98
1.2 Understand meanings of operations and how they relate to one another	pp. 24, 28, 32, 36, 48, 50, 52, 56, 60, 62, 64, 68, 72, 82, 86, 88, 90, 94, 98
1.3 Compute fluently and make reasonable estimates	pp. 24, 28, 32, 36, 40, 44, 46, 48, 50, 52, 56, 60, 62, 64, 68, 72, 82, 86, 88, 90, 94, 98

Algebra	
2.1 Understand patterns, relations, and functions	pp. 24, 28, 36, 44, 46, 48, 50, 52, 72, 74, 78
2.2 Represent and analyze mathematical situations and structures using algebraic symbols	
2.3 Use mathematical models to represent and understand quantitative relationships	
2.4 Analyze change in various contexts	

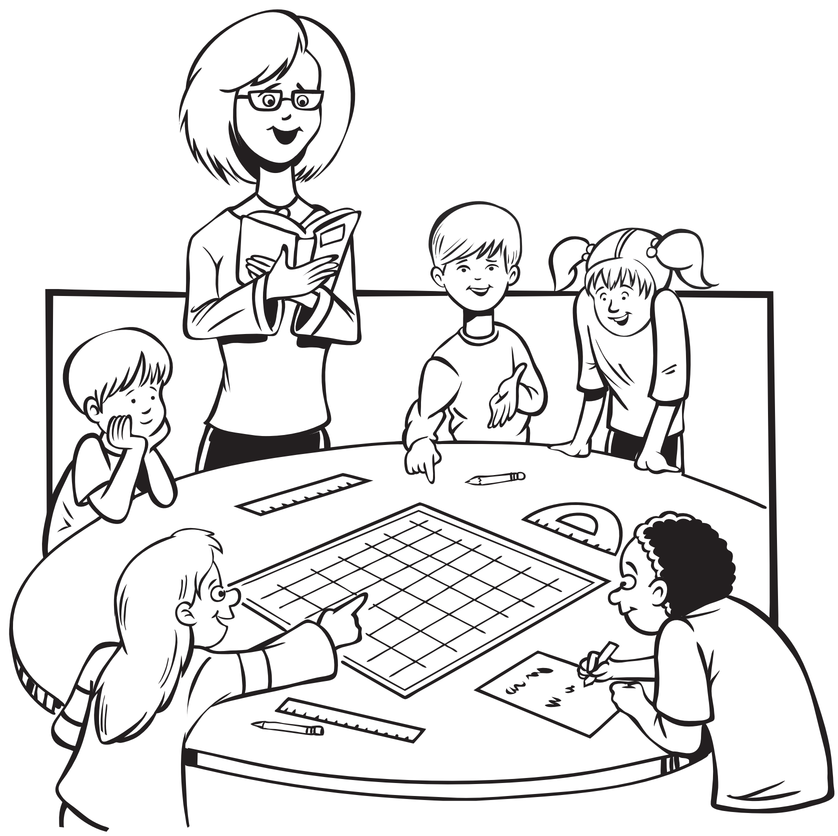
Geometry	
3.1 Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships	pp. 74, 78, 82
3.2 Specify locations and describe spatial relationships using coordinate geometry and other representational systems	pp. 74, 78, 82, 86, 88, 90
3.3 Apply transformations and use symmetry to analyze mathematical situations	pp. 74, 78, 82
3.4 Use visualization, spatial reasoning, and geometric modeling to solve problems	pp. 74, 78, 82

Data Analysis and Probability	
4.1 Formulate questions that can be addressed with data, and collect, organize, and display relevant data to answer them	pp. 24, 32, 36, 48, 50, 52, 56, 60, 62, 64, 68, 72, 90, 98
4.2 Select and use appropriate statistical methods to analyze data	pp. 94, 98
4.3 Develop and evaluate inferences and predictions that are based on data	pp. 94
4.4 Understand and apply basic concepts of probability	pp. 94

Problem Solving, Reasoning and Proof, Communications, Connections, Representation	all activities

Math Problem-Solving Skills:

Teacher Notes and Student Worksheets



Problem-Solving Objective

To interpret and organize information in a series of interrelated statements and to use logical thinking to find solutions

Materials

Calculator

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Data Analysis and Probability 4.1

Focus

These pages explore the concepts of averages, distances, and payments and require students to interpret information found in a series of interrelated statements. Students must read the problems carefully and consider a number of different criteria. Tables and lists can be used to help manage the various criteria.

Discussion

Page 25 – How Many?
Analysis of these investigations involves a series of interrelated amounts that must be read carefully to find a solution. The use of a table or list could be very helpful to manage the data. For Problem 2, a table listing the years 2003 to 2007 could be used as a starting point. The problem states the number of visitors in 2004, and this information can be used to figure out the number of visitors in 2007— twice as many. That information can in turn be used to figure out the amounts for the other years. A similar table or list can be used for the other problems. The problems contain additional information that is not needed to find a solution.

Page 26 – How Far?
These problems explore the concept of average distance traveled over a period of time. In many cases the solution is not necessarily exact but rather an approximate time or distance; for example, Problem 1 states that Kelly runs 100 meters in "about 40 seconds." This is not an exact time and would vary from lap to lap, so the answer would be an approximate distance. Problem 4 contains information about morning coffee and lunch that must be factored into the amount of time spent driving. If they leave at 9:30 in the morning, stop for 30 minutes for coffee, and then have lunch at 12:30, they have driven for two and a half hours and not three hours.

Page 27 – How Much?
Students must read each problem carefully to determine what the problem is asking, since some investigations have information that is not needed to find a solution. The first investigation requires students to calculate how many 4-ounce packs can be made from 68 pounds of cheddar cheese and use this to solve how much profit is made in a year. The information about Swiss cheese is not required.

In the problem about the doors and hinges, it is important to include the door price in the solution, since it asks for the cheapest option that would include a door. Students must calculate the cost of two hinges with screws included and two hinges with screws bought separately, since the cost of the door remains the same for both options.

When exploring the problem about the melons, a table can be used to manage the data showing the various multiples and the profit associated with each multiple. This in turn can be used to explore a pattern to find the solution.

Possible Difficulties

- Not using a table or list to manage data
- Not understanding the concept of averages
- Confusion when dealing with approximate times and distances

Extension

- Write other problems using the same form of complex reasoning for other students to solve.

1.1 HOW MANY?

1. About 1.6 million people visit the Great Barrier Reef each year. During September, 228,000 tourists visited the reef, while there were twice as many visitors in May as in February, but 6,000 fewer than August. August had 9,000 more visitors than September. How many visitors were there in February?

2. In 2007, a record number of tourists visited Denali National Park in Alaska. There were twice as many visitors in 2007 than in 2004. There were 17,660 more visitors in 2007 than in 2006, and 36,180 fewer visitors in 2005 than 2006. 2004 and 2003 had similar numbers, as there were only 5,790 more in 2004 than in 2003. If there were 192,980 visitors in 2004, how many were there in 2006?

3. About 815,200 people visit Tasmania, an island off the coast of Australia, each year. Visitor numbers are highest during the warmer months of October to March and lowest during the colder months of May to August. During December, there were 86,593 visitors. November had 46,474 more visitors than June, and July had 45,219 fewer than December. June had 63,866 fewer visitors than January, while January had 57,455 more visitors than July. How many visitors were there in November?

4. Last year, just over 500,000 people visited Carlsbad Caverns in New Mexico. Currently, tours are available for five of the caves. Over 80% of visitors to the park tour either King's Palace, Left Hand Tunnel, Lower Cave, or Hall of the White Giant, as well as visiting the main cave. During September, 45,470 people visited, while October had 1,570 more than November and November had 2,390 fewer than August. August had 1,620 more visitors than September. How many visitors were there in October?

1.2 HOW FAR?

1. Kelly runs around a 100 m track each day of the week and around a 200 m track on the weekend. She averages 100 m in about 40 seconds. She usually runs for 50 minutes each day during the week and an hour each day on the weekend. Approximately how far does she run each week?

   ____

2. Miranda caught the bus from Kansas City to Little Rock. The bus left at 8:20 a.m. and, due to traffic, averaged 29 miles per hour for the first 80 minutes. Once on the highway, the average speed increased to 59 miles per hour. The bus stopped for lunch at 12:15 p.m. How far had Miranda traveled?

   ____

3. Yin trains each day for the triathlon. During the week, she runs 5 km each morning and swims 3 km each afternoon. On the weekend, she rides 12 km on Saturday and 16 km on Sunday. What is the total distance she runs, swims, and rides each week?

   ____

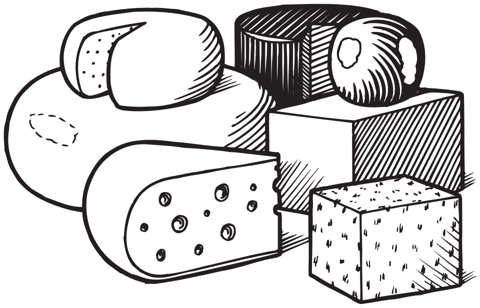
4. Simon and his brother drove from San Diego to San Francisco, a distance of 500 miles. They left at 9:30 in the morning and had lunch at 12:30. They stopped for 30 minutes for coffee and an hour at lunch. They averaged 54 miles per hour before lunch and 62 miles per hour after lunch. How far had they driven by 3:00 p.m.?

   ____

5. Denver is about 1,020 miles from Chicago. The train from Chicago to Denver travels at an average speed of 77 miles per hour. If the train travels directly without any stops, how far will it have traveled after 7 hours and 25 minutes?

   ____

1.3 HOW MUCH?



1. Each month, Kofi's cheese shop sells 68 lb of cheddar cheese and 36 lb of Swiss cheese. He buys the cheddar cheese in 10 lb blocks for $96.50 each and the Swiss cheese in 3 lb rounds for $37.00 each. He then divides the cheese into smaller 4 oz packages, which he sells for $4.50 each. He discards unsold cheese at the end of each month. How much profit does he make on cheddar cheese each year?

____

2. Madeline delivers advertising leaflets. She can be paid by the number of leaflets she delivers or by the hours she works. The rate per leaflet is five and a half cents, and the hourly rate is $4.25 for the first hour and $3.50 for the other hours. Which option pays more money if she delivers 539 leaflets over six and a half hours?

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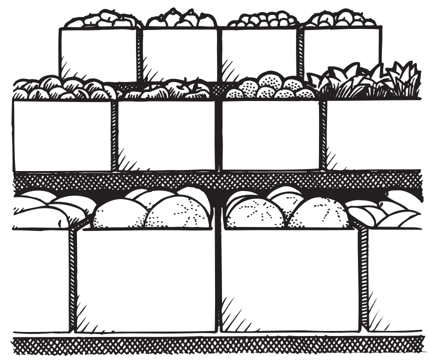


3. Courtney bought melons from the market at 6 for $5 and sold them at her fruit store at a price of 3 for $4. If she made a profit of $84, how many melons did she sell?

____

4. Bill needs a new front door. The hardware store sells doors and hinges with and without screws. Hinges with screws cost $24.90 each, and the ones without screws cost $20.50 each. Door prices start at $149.00, and screws start at $0.80 each. Each door needs two hinges, and each hinge needs four screws. Which is the cheapest option for Bill to buy a door, and by how much?

____

5. Daniela bought some new towels for her inn. She paid $72 for the towels at a cost of $9 each. The next day she saw the same towels for $7 each, so she bought twice as many as the day before. How much did she spend in total on the towels?

____

Problem-Solving Objective

To analyze and use information in word problems

Materials

Calculator

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Algebra 2.1
- Data Analysis and Probability 4.1

Focus

These pages explore word problems that mostly require multiplication or division. Students must determine what the problem is asking and in many cases carry out more than one step to find a solution. Analysis of the problems reveals that some questions contain additional information that is not needed. If necessary, a calculator can be used to assist with the calculation, since these problems are about determining what the problem is asking rather than computation or basic facts.

Discussion

Page 29 – The Seedling Nursery

In most cases each problem requires more than one step and involves multiplication. Problem 3 involves the concept of profit, while the last investigations have a number of combinations and students can use various methods to solve them. One way to solve Problem 5 would be to look at multiples of 4 and 9 and put them into a table to find the various combinations and then look for a pattern regarding the multiples.

Multiples of 4	Possible multiples of 9	Total
4	234	238 – too few
8	225	233 – too few
…	…	…
32	207	239

Page 30 – The Tropical Orchard

The information provided in the beginning statement is needed to answer the subsequent questions. Using the original information as a basis, the numbers are changed to meet new criteria as mangoes and bananas are sorted and packed into tray and cartons. Care is needed, since mangoes are packed according to number and bananas are packed according to weight. Some solutions may not necessarily be exact; for example, 5,664 mangoes were packed into trays. If 6 per one hundred are rejected, then there were 56 hundreds with 6 mangoes rejected and part of a hundred (64). This could be written as "about 57 hundreds" or as "56 hundreds and about two-thirds of a hundred," giving solutions of 57 × 6 (to give 342) or 56 and $\frac{2}{3} \times 6$ to give a more accurate approximation of 340.

Page 31 – Animal Safari Park

These problems require more than one step of calculations and involve a number of operations, including multiplication and division. The wording has been kept simple to assist with the problem-solving process. Some problems have added information (often involving weight) that is not needed to find a solution, and some problems may not have an exact answer; for example, Problem 2 results in 53 boxes of fish—if the penguins eat 3 boxes a day, then there is enough fish for 17 full days and 1 part day (two boxes). Students could discuss their solution in terms of 17 days and two-thirds of a day. Students should be encouraged to explore different ways of arriving at a solution.

Possible Difficulties

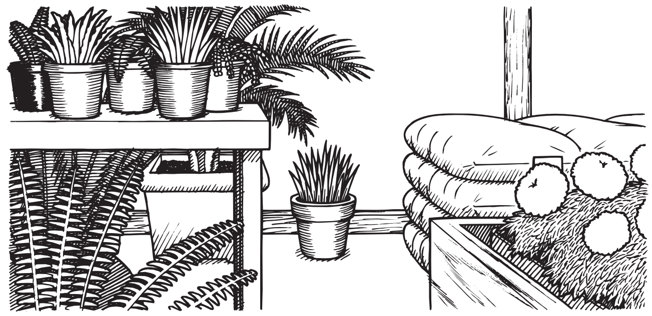
- Confusion over the need to carry out more than one step to arrive at a solution
- Using all of the numbers listed in the problems rather than just the numbers needed
- Difficulty with the concept of profit

Extension

- Discuss how problems can have more than one answer depending on different interpretations.
- Students could write their own problems and give them to other students to solve.
- Explore how many of these problems could be solved using the repeated addition technique on the calculator.

2.1 THE SEEDLING NURSERY

1. During the week, the nursery planted 258 flower trays and 87 fern trays. A flower tray holds 8 seedlings and a fern tray holds 6 seedlings. How many seedlings were planted during the week?



____

____

2. During the morning, 134 flower trays were fertilized and 120 fern trays were watered. During the afternoon, another 56 trays were watered and 48 trays were fertilized. If there are 6 seedlings in each of these trays, how many seedlings were watered?

____

3. (a) The nursery sold 134 bags of bark mulch over 7 days. If each bag sells for $29, how much money did the nursery receive from selling bark mulch?

____

(b) If each bag of bark mulch costs the nursery $11 to produce, how much profit did the nursery make from the bark mulch?

____

4. During the morning, the nursery used trays that held 4, 6, or 8 seedlings. How many seedlings were planted if 115 small trays were used?

____

5. During the afternoon the nursery used trays that held either 4 or 9 seedlings. If 239 seedlings were planted, what combination of trays could have been used?

____

6. The next day, the nursery used trays that held either 6 or 8 seedlings. If 376 seedlings were planted, what combination of trays could have been used?

____

2.2 THE TROPICAL ORCHARD

The tropical fruit farm has 14 hectares of mangoes and 6 hectares of bananas.

1. Each hectare of mangoes has 94 trees and is irrigated on a rotation basis. Two and a half hectares are watered each day before sunrise. How many trees are watered each day, and how many days would it take for all of the trees to be watered?

____

2. All of the bananas are irrigated over a 3-day period. If each hectare has 1,450 banana plants, how many hectares are watered each day?

____

3. (a) In the packing shed, mangoes are sorted and placed into trays that hold 16 mangoes each. Meanwhile, bananas are sorted into 13 kg cartons. When the truck arrives to transport the fruit to the market, there are 198 cartons and 354 trays ready to be transported. How many mangoes are packed and ready to go?

____

(b) Some damaged and blemished mangoes are rejected and are not packed into trays. About 6 mangoes per 100 are rejected. Approximately how many mangoes were rejected during the day?

____

(c) How many kilograms of bananas were packed during the day?

____

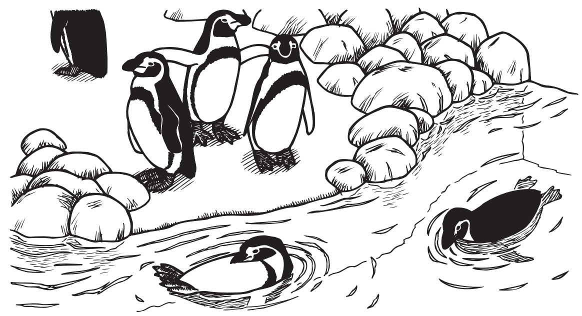
4. After 3 years, new banana plants are needed. Each year some plants are removed and replaced with new plants so that each plant is replaced over a three-year cycle. How many new plants are replaced each year?

____

2.3 ANIMAL SAFARI PARK

1. At Animal Safari Park, the tropical birds in the walk-through aviary eat 7.5 bags of seed a week. Each bag weighs 40 kg and costs $63. How many bags are used over one year?

2. The penguins eat 3 boxes of small fish a day. Each box weighs 2.4 kg and costs $6.50. The park has just purchased $344.50 worth of fish. How long will this last the penguins?



3. The elephants eat 7 bales of hay over 2 days. Each bale of hay costs $12. How much would it cost to feed the elephants during the month of July?

4. Seals eat 5 cartons of medium-sized fish a month. Each carton weighs 12 kg, contains 6 boxes of fish, and costs $57. How many kilograms of fish do the seals eat over a year?

5. The parking lot has 12 sections for cars to park in. Each section holds up to 176 cars. If sections 1–6 are completely full and sections 7–9 are half full, how many cars are in the parking lot?

6. The area to see the birds of prey show has 23 rows of seats, with 27 seats in each row. If it is two-thirds full, how many more people can watch the show?

7. The cafeteria seats 250 people at tables inside and 65 people outside. Currently, there 23 vacant seats inside and 16 vacant seats outside. How many people are seated?

Problem-Solving Objective

To analyze and calculate information in written numerical problems

Materials

Calculator, if necessary

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Data Analysis and Probability 4.1

Focus

These pages explore word problems that require a number of operations, including division. The wording has been kept fairly simple to help with the problem-solving process. Students must determine what the problem is asking and, in many cases, carry out more than one operation to find solutions. If necessary, a calculator can be used to assist with the calculations, since these problems are about reading for information and determining what the problems are asking rather than computation or basic facts.

Discussion

Page 33 – At the Mall
These problems focus on discount and sale prices. Students must have an understanding of the concept of how the marked price of goods can be discounted. Some problems involve calculating the price after the discount has been applied, while others involve calculating the full price of a product as compared to its sale price. A table can be used to help with the problems that have a variety of possible answers. A calculator can be used assist if necessary—key in the full amount, press the subtraction key, enter in the discount, press the percent key, and the sale price will displayed.

Page 34 – At the Deli
These investigations require more than one step and involve concepts of grams and kilograms and ounces and pounds. The concepts of profit and repackaging are also explored in many of the problems. Students may find it helpful to draw diagrams to visualize what is happening. A table can be used in the first investigation to help manage the data, while a diagram may be helpful in Problem 2 to visualize the actual process of repackaging and the cost involved in buying the large tubs and selling the smaller containers.

Page 35 – The Sugar Mill
The sugar mill problems involve the concept of metric tons. In most cases more than one step is needed to find a solution. The first investigation about metric tons of cane over a season results in a calculation of about 815.8 bins. This is an example of an answer that does not make sense in light of the problem and must be thought of as 815 full bins and one bin that is not full, or 816 bins. Similar thinking is required for most of the problems. The problem about the two farms results in approximately 2,123 metric tons per month, which would be about 1,061.5 metric tons per farm. The numbers are not exact and can be used to discuss how totals such as these are calculated and in most cases would not be exact, but rather approximations. The last problem sees 353 truckloads, so if 4 trucks were used, there would be 88 trips using 4 trucks and 1 trip using 1 truck.

Possible Difficulties

- Confusion over the need to carry out more than one step to arrive at a solution
- Using all of the numbers listed in the problem, rather than just the numbers needed
- Not thinking in terms of the problem and writing solutions such as 6.04 trains
- Difficulty with the concepts of metric tons, discount, percent, and profit

Extension

- Students could write their own problems and give them to other students to solve.

3.1 AT THE MALL

1. (a) The computer store has a sale of 25% off the marked price of all desktop and laptop computers. I bought a scanner with a marked price of $149 and a desktop computer with a marked price of $1,299. How much did I pay after the sale price had been taken into account?

   (b) Later that day, one of my friends bought a new laptop computer and a laser printer for $1,871.25. If the marked price of the printer was $375, what was the marked price of the laptop?

2. (a) The music shop has 10% off the marked price of all DVDs and 15% off the marked price of all CDs. I bought 2 DVDs, which had a marked price of $15 each, and 4 CDs, which had marked price of $25 each. How much did I pay?

   (b) My friend also bought some DVDs and CDs. At the checkout he paid $83. What could he have bought?

3. (a) The department store is selling T-shirts at $19.99 each or 3 for $50. It is also selling cargo shorts at 10% off the marked price of $25. If I spent $145, what could I have bought?

   (b) Since I spent over $100, I was eligible for a store discount card that would give me 5% off all future purchases. How much would I have spent if I could have used the card on my previous purchases?

   (c) My friend bought 3 T-shirts and 4 pairs of shorts. How much did he spend?

3.2 AT THE DELI

1. The deli sells tomato sauce in both jars and cans. 16 boxes of tomato sauce have been delivered. Nine boxes contain cans and 7 boxes contain jars. Each box has either 12 jars or 16 cans of tomato sauce. Eight boxes have been unpacked, and there is a total of 116 cans and jars ready to go on the shelves. How many boxes of jars were unpacked?

2. (a) The deli buys grilled eggplant in 4 lb tubs, which they repackage and sell in 8 oz containers. If each tub costs them $17 and they sell the containers for $4, how much profit do they make on each tub?

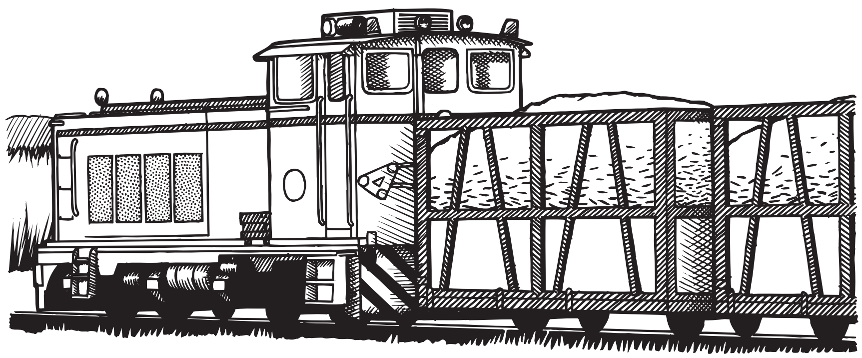
   (b) About 26 containers are sold each week. How many tubs does the deli need to buy each week so that there are enough containers to sell?

   (c) How much profit would the deli make selling eggplant during the month of February? (Unsold eggplant is discarded at the end of each week.)

3. (a) The deli sells marinated feta cheese in 400 g tubs and in bulk by the kilogram. The tubs sell for $9.50 and the bulk feta for $24.50 per kg. Which is the cheaper way to buy 1.2 kg of feta?

   (b) The deli buys its bulk feta for $15.30 per kg. During the summer months, about 27 kg of feta are sold each week, and during the winter months, about 13 kg are sold. What is the difference in profit per week between the summer and winter months?

3.3 THE SUGAR MILL



During the crushing season, harvested sugar cane is transported to the mill by rail.

1. The farm closest to the mill sent 4,895 metric tons of cane to the mill. A train usually hauls between 135 and 145 bins. If each bin carries 6 metric tons of cut sugar cane, how many bins of cane were transported?

2. If trains usually haul a maximum of 145 bins and a minimum of 135 bins, what would have been the maximum and minimum number of trains used by the cane farm during the season?

3. Two brothers run adjoining farms. To save on costs, they transport their cane to the mill as a combined haul. Last season they transported 14,860 metric tons of sugar cane to the mill. If harvesting season extends from June to December, approximately how much cane was harvested per month per farm?

4. (a) During one week, trains transported cane from 6 different farms. If 29,100 metric tons of cane were transported, how many bins and trains were used?

   (b) It takes approximately 8 metric tons of cut sugar cane to produce 1 metric ton of raw sugar. How much raw sugar would be produced by the 6 farms?

5. During the crushing season, raw sugar from the mill is transported to the port by trucks, each of which can carry 7 metric tons. How many trips would be needed to transport 2,470 metric tons of raw sugar if 4 trucks were used?

Problem-Solving Objective

To read, interpret, and analyze information

Materials

Calculator, if needed

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Algebra 2.1
- Data Analysis and Probability 4.1

Focus

These pages explore concepts of place value, number sense, and using data. The relationships among numbers and place value are analyzed and students are encouraged not only to find possibilities but also to disregard numbers and combinations that are not possible. Data must be interpreted and analyzed to find solutions.

Discussion

Page 37 – Bookworms

Since the book has 18 chapters of equal length, a calculator may be used to easily calculate how many pages are in each chapter. This information can then be readily used to calculate the last page of each chapter and the starting page of the following chapter. Before beginning, students may find it helpful to draw up a table outlining the starting page, the halfway point, and the last page of each chapter. We know that each chapter has 124 pages and that Chapter 1 starts on page 1, so Chapter 2 must start on page 125, and so on. Entering 124 + 124 and pressing the = key on the calculator repeatedly will give the last page of each of the 18 chapters.

The page number that Nathan has read to is already provided, and this information is needed to answer some of the questions. For example, to determine about how many pages Nathan has read past the middle of the book, it is necessary to keep in mind that there are 2,232 pages in the book and that he has already read up to page 1,674. Similarly, this information is needed to determine how many pages he must read to finish the book.

Page 38 – Profit and Loss

In these investigations, students must think in terms of fixed costs as those occurring regardless of income and variable costs as those occurring according to production. The variable cost per item is not given as such, and students must use the information that 500 items have a variable cost of $7,500 to identity that the variable cost per item is $15.

Once the table is complete, the data can be used to determine at what point the factory starts making a profit. The point at which this occurs is not shown in the table, and students must take what they know—that the factory is showing a loss at 100 items but making a profit at 250—to see that the break-even point will occur somewhere in-between.

Page 39 – Calculator Patterns

These problems are easy to state and investigate using a calculator's memory function to simplify the steps and assist in developing algebraic thinking, since the patterns described in words in the examples used are translated into early proofs in general. The first problem investigates a pattern based on the difference of two squares, with a focus on why the result would be true in general. The second problem is an investigation based on the difference of two cubes. The pattern can be discerned as the examples are worked through using the calculator. However, showing why this is true in general requires quite sophisticated algebraic reasoning that students should understand but not necessarily arrive at on their own.

Possible Difficulties

- Difficulty with the concepts of fixed cost and variable costs
- Difficulty with the concepts of profit and loss
- Poor understanding of place value
- Wanting to add, subtract, or multiply rather than using place value or number sense
- Not using all the criteria

Extension

- Students could think up their own profit and loss problems using different criteria.

4.1 BOOKWORMS

Nathan's book starts on page 1 and has 2,232 pages. There are 18 chapters of equal length in the book, and he has read up to page 1,674.

1. How many pages are in each chapter?

2. Nathan's favorite part of the story is on page 874. What chapter is it in?

3. Nathan reads half a chapter each day. How long has he been reading the book?

4. Nathan's favorite chapter is Chapter 11. What pages are in Chapter 11?

5. The most exciting part was from page 1047 to page 1149. In what chapters are these pages?

6. How many pages past the middle of the book has Nathan read?

7. Nathan's friend is also reading the same book. He has read 14 pages of Chapter 8. What page is he up to?

8. How many more pages must Nathan's friend read to be where Nathan is?

9. How many more pages must Nathan read to finish the book?

10. How many more days will it take for Nathan to finish the book?

4.2 PROFIT AND LOSS

A factory owner knows that some of her expenses are going to occur regardless of how many items she makes and sells. These are fixed costs and include things such as rent, insurance, phone, and Internet. Her fixed costs total $4,000 per week.

Her other expenses are variable costs, which change each week depending on the number of items she makes. Variable costs include things such as materials, labor, and electricity.

All factory items are made to order, so all items produced are sold. In one week she sold 500 items, and her variable costs were $7,500. All items sell at $50 each.

1. Complete the table below to show the income for the factory based on the number of items sold, as well as the total costs for each week.

Items sold per week	50	75	100	250	500	1,000	1,500	3,000
Income								
Total costs								

The difference between total costs and income is the factory owner's profit. If she made no sales, she would still have to pay the fixed costs and would have a loss for that week.

2. Did she make a profit or a loss when she sold 100 items?

____

3. Did she make a profit or a loss when she sold 1,000 items?

____

4. Using the information in the table, estimate the number of items she needs to sell each week to break even. Calculate the expenses and income for that number of items to check your estimate.

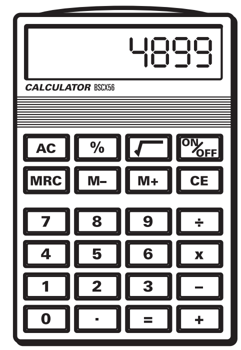
____

4.3 CALCULATOR PATTERNS

Use your calculator to help you think about what is happening.

Squaring and Multiplying

1. Choose 3 consecutive 2-digit numbers—for example, 69, 70, 71.
   (a) Multiply the first and third numbers.
   (b) Square the middle number.
   (c) What do you notice?
   (d) Try some other 2-digit numbers.
   (e) Try some 3-digit numbers and 4-digit numbers.
   (f) Describe the pattern. Why does this happen?



Cubing, Squaring, and Multiplying

2. Choose a 2-digit numbers—for example, 47.
   (a) Make a new number by finding the difference between the cube of the tens digit and the cube of the ones digit.
   (b) Make another new number by adding the square of the tens digit, the square of the ones digit, and the product of the tens and ones digits.
   (c) Divide the first new number by the second new number.
   (d) Try some other 2-digit numbers.
   (e) What do you notice?

Problem-Solving Objective

To identify and use number understandings

Materials

Calculator, if necessary

NCTM Content Standards

- Number and Operations 1.3

Focus

These pages ask students to use number sense and logic to solve problems involving magic squares and sudoku and alphametic puzzles. Analysis of the problems to locate given information is necessary to find the magic number or the arrangement of numbers. Counters, blocks, or a calculator may be used to assist, since these problems focus on number sense and logic rather on than basic facts.

Discussion

Page 41 – Magic Squares
This investigation involves the concept of magic squares. All rows, columns, and diagonals in a magic square add to the same total. 3-by-3, 4-by-4, and 5-by-5 magic squares have been used.

Page 42 – Sudoku
This page explores the concept of sudoku. The word *sudoku* roughly means "digits must occur only once." In this case, 6-by-6 grids have been used, so every row, column, and minigrid must contain one of each of the digits 1–6. No addition or basic facts are required, and students must use logical reasoning to find solutions.

Page 43 – Alphametic Puzzles
The puzzles on these pages are known as alphametic puzzles or cryptarithms, meaning that letters in words are substituted for numbers in an addition algorithm. There are a number of famous alphametic puzzles, such as "send more money" and "no more cash." In the example provided, the S must be a 1, since the equation involves adding a four-digit number and three-digit number and, as such, can only result in a 1 in the tens thousands place. As the M and C in the second puzzle are different letters, the O must be a 9 in order for them to be different letters. This gives the ones digit in the word *no*. Some of the alphametic puzzles have a number of different mathematical equations that fit the criteria. Students should be encouraged to see how many different possibilities there are.

Possible Difficulties

- With the magic squares, considering only rows or columns rather than rows, columns, and diagonals
- Not thinking strategically when doing the sudoku or the alphametic puzzles

Extension

- Investigate other magic squares, magic numbers, and alphametic puzzles.
- Explore sudoku games in magazines, newspapers, and on the Internet that involve 6-by-6 grids as well as 9-by-9 grids.
- Try writing other alphametic puzzles for other students to solve.

5.1 MAGIC SQUARES

Magic squares have rows, columns, and diagonals that all add to the same total.

15	1	11
5	9	13
7	17	3

1. This magic square has a magic number of ___________.

2. Complete the magic squares. Remember, all rows, columns and diagonals must add to the same number.

(a)

16		
	20	28
		24

Magic number ___________

(b)

	3	
15	27	
	51	

Magic number ___________

(c)

	35	
56	7	42

Magic number ___________

(d)

75		55
	45	
		15

Magic number ___________

3. Complete these 4-by-4 magic squares.

(a)

	16	10	
	40	24	
	6		26
18	28		8

Magic number ___________

(b)

30		12	
	48		18
		27	36
24	39		9

Magic number ___________

(c)

28	22		46
18		30	
	12		32
24			14

Magic number ___________

This is a fifth-order magic square. It has 5 rows and 5 columns.

4. Complete the magic square and find the magic number.

Magic number ___________

9		3	15	22
	12		6	
		20	2	14
17	4			10
13		7	19	

5.2 SUDOKU

Sudoku puzzles are made up of numbers. To solve them, you must use logic to figure out where the numbers go.

Every row, column, and minigrid must contain one of each of the numbers 1, 2, 3, 4, 5, and 6.

4	5	2	6	3	1

This **row** has the digits 1, 2, 3, 4, 5, and 6

					1
					2
					3
					4
					5
					6

This **column** has the digits 1, 2, 3, 4, 5, and 6

			5	3	1
			6	4	2

This **mini-grid** has the digits 1, 2, 3, 4, 5, and 6

4	5	2	6	3	1
3	1	6	5	4	2
5	2	3	1	6	4
6	4	1	3	2	5
1	3	4	2	5	6
2	6	5	4	1	3

1. Complete each sudoku using the digits 1 to 6.

(a)

6				3	
	3		2		
	2		4	1	6
1		6			
4		3		5	
	5				3

(b)

			1	4	
	4	2		6	3
	6	5			4
			6	3	
2	1			5	
	5	4			1

(c)

		2	1	9		8	3	
8	3			6	2	5		4
4				3		7		1
	1	8	6	2	5			7
	6		8		3	9		2
			9	1	7	6		
1	4	9		5				8
6				7			5	9
	5	7			9	1		

(d)

8	9		5		1		3	
6		4		7	2	8	5	9
		2		8				7
1		8		2	3		9	
	4				9			
	7		1	5		6		3
4				3	7	9		
	8	6		1		3	7	
	3	1	6		4			2

5.3 ALPHAMETIC PUZZLES

An alphametic puzzle is an arithmetic problem involving words where there is a one-to-one swapping between letters and digits that makes the arithmetic correct. These puzzles are also called "cryptarithms."

In the example below, you can see that the phrase "days too short" becomes an arithmetic algorithm. (Addition is used.)

		D	A	Y	S
+			T	O	O
	S	H	O	R	T

		9	8	7	1
+			6	5	5
	1	0	5	2	6

1. There are many famous and well-known alphametic puzzles, some of which are listed below for you to solve. (These all use addition.)

(a)

		S	E	N	D
+		M	O	R	E
	M	O	N	E	Y

		5		
	1		8	
				2

(b)

			N	O
+	M	O	R	E
	C	A	S	H

			9
2		3	

(c)

				Y	O	U
+	S	I	L	L	Y
	G	O	O	S	E

			0	
	9	2		
				5

(d)

			D	O
		Y	O	U
	F	E	E	L
L	U	C	K	Y

			5	
	0			8

Problem-Solving Objective

To use patterns and logical reasoning to determine numbers in spatial arrangements

Materials

Counters in two different colors, calculator

NCTM Content Standards

- Number and Operations 1.3
- Algebra 2.1

Focus

This page explores students' understanding of numbers in order to discern patterns that allow larger numbers to be determined without laboriously writing or counting all of the numbers up to the point asked for. It also relates to the number patterns seen in square numbers.

Discussion

Page 45 – Number Patterns 1
Students should use different-colored counters to explore the pattern shown on the page and extend it to larger numbers to describe the relationship between the counters and the square. The sum of the first two odd numbers, 1 + 3, is 4, or 2^2; 1 + 3 + 5 is 3^2; 1 + 3 + 5 + 7 is 4^2; and so on. The sum of the first 6 odd numbers is 6^2, or 36, and the sum of the 10 first odd numbers is 10^2, or 100. Other patterns can also be seen with the square numbers shown this way.

This arrangement of numbers in a triangular pattern extends from previous problems in which numbers were placed in columns . It also links to the square numbers and odd numbers examined in the first problem. There is 1 number in row 1, 3 numbers in row 2, 5 numbers in row 3, 7 numbers in row 4—all of which are the 1st, 2nd, 3rd, and 4th odd numbers. The next row should have 9 numbers, and so on. Also, further observation of the pattern of the numbers shows that the numbers at the end of each row are square numbers. Putting these two pieces of information together allows the position of any number to be found. Linking this to the first problem shows that the end number of each row is also the amount of numbers to that point.

Finding the other numbers is equivalent to finding the nearest square number and then seeing how the rows add one number to the beginning and one to the end each time.

Possible Difficulties

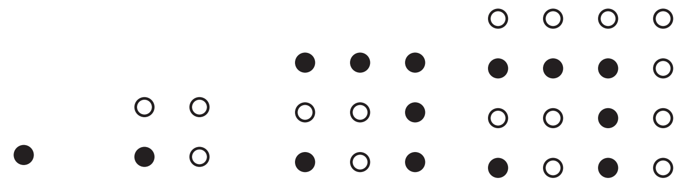
- Thinking that writing out all of the numbers is the only way to be sure of a solution
- Unable to see how the odd and square numbers link to the patterns
- Unable to verbalize a mathematical description of how the odd numbers and square numbers relate to the placement of the numbers in the triangular pattern

Extension

- Examine what would happen if only even numbers were placed in the triangular pattern.
- Challenge students to find a relationship between the triangular and square number patterns.
- Find some background information about Pythagoras, an Ancient Greek mathematician who was interested in number patterns and geometry.

6.1 NUMBER PATTERNS 1

Ancient Greek mathematicians were interested in numbers made from different arrangements of counting objects. Numbers arranged in a square pattern (for example, 3-by-3, 4-by-4) gave the sum of the odd numbers.



1. What is the sum of the first 4 odd numbers? ______________

2. (a) What is the sum of the first 6 odd numbers? ______________

   (b) 10 odd numbers? ______________

3. Can you find a relationship between the numbers arranged in the patterns and the sum of the odd numbers? If so, describe in words how you think it works.

   ____

Examine the counting numbers when arranged in the shape of a triangle.

1
2 3 4
5 6 7 8 9
10 11 12 13 14 15 16
17 18 19 …

4. What number would be at the end of the 5th row?

   ____

5. What number would be at the end of the 12th row? ______________

6. What do you notice about how the amount of numbers in each row increases?

   ____

7. What do you notice about the numbers at the end of each row? ______________

8. Where would 101 appear? ______________

9. In what row would 183 appear? ______________

10. What number would be directly below 168? ______________

11. Is there any relationship between the arrangement of the odd numbers in the squares and how they are arranged in the triangle. If so, what is it?

    ____

Problem-Solving Objective

To use patterns and logical reasoning to determine numbers in spatial arrangements

Materials

Counters in two different colors, calculator

NCTM Content Standards

- Number and Operations 1.3
- Algebra 2.1

Focus

This page explores the understanding of numbers to discern the patterns found in triangular numbers. It links to the square numbers patterns investigated in the previous activity. The use of letters to summarize patterns lays a foundation for the thinking used in algebra.

Discussion

Page 47 – Number Patterns 2

Students should use different-colored counters to explore the pattern shown on the page and extend it to larger numbers in order to describe the relationship between the counters and the triangle. The sum of the first two numbers, 1 + 2, is 3. The sum of the first three numbers, 1 + 2 + 3, is 6. The sum of the first four numbers, 1 + 2 + 3 + 4, is 10, and so on. Two of the same triangular numbers give the corresponding square number plus the number, showing a given triangular number is half of [number^2 + number]. Students should use their counters to investigate how this pattern continues for larger triangular numbers, introducing the use of T_1, T_2, T_3, … Another pattern that students might see is that the triangular number is half [number × (number +1)].

When 2 consecutive triangular numbers are added, they give the square number corresponding to the larger triangular number. This is readily shown with counters for a series of numbers and can be expressed as $T_2 + T_3 = S_3$, $T_3 + T_4 = S_4$, and so on.

This arrangement of numbers in a triangular pattern, used by Pascal and first suggested by ancient Chinese mathematicians, is very helpful for summarizing relationships in probability (Pascal) and algebra (Ancient China). Many interesting patterns can be discerned among the numbers. The outside diagonals of the triangle consist only of 1, the next diagonal has the counting numbers, and the triangular numbers are in the third diagonal, beginning with 3 (T_2). When the triangular numbers are summed, their sum is diagonally below the last number.

Many other patterns can be investigated as the triangle is extended. Have students write out the numbers on a sheet of paper to get as large a triangle as they can. Ask them to highlight the triangular and square numbers in the triangle. Ask them to explore questions such as:

- Is it possible to have a square triangular number?
- Is there a pattern to the location of the square numbers?
- Can they find a way to describe a pattern for the numbers in the other diagonals?

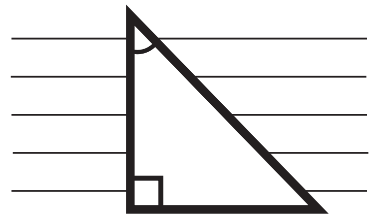
Possible Difficulties

- Students may find it difficult to accept and use the algebraic form of notation involving T_2 or S_2, and so on.
- Unable to see how two triangular numbers form a pattern based on a square number plus the number or as a rectangular pattern of the number × [number + 1]
- Unable to complete the pattern to give Pascal's triangle

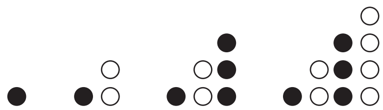
Extension

- Investigate other arrangements of counters to give numbers; these forms are called polygonal numbers and extend to pentagonal and hexagonal numbers, and so on. Do any of these numbers occur on the Pascal triangle?
- Is it possible to find a relationship between the triangular or square numbers and other polygonal numbers?
- What would happen if this triangular pattern began with the number 2 instead of 1?
- Find some background information about Pascal, the French mathematician who used the triangular patterns of numbers that were later named in his honor.
- Investigate the history of this triangle from the times of the Chinese mathematicians and the way it is used currently in mathematics and in applications.

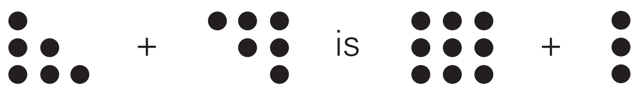
7.1 NUMBER PATTERNS 2



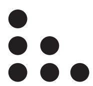
Ancient Greek mathematicians were interested in numbers made from different arrangements of counting objects. Numbers arranged in a triangular pattern suggested the sum of the counting numbers.



1. Putting 2 of the same triangular numbers gave the corresponding square number + that number; for example:

   + is + or $3^2 + 3$

   is half of [$3^2 + 3$]

   The third triangular number, T_3, is half of [$3^2 + 3$].

   (a) Is this true for the 4th triangular number, T_4? T_5? T_6? … ____________

   (b) What happens when you add a triangular number and the next triangular number?

   ____

   (c) Show why this is so using a diagram like the one above.

   (d) Can you write this pattern using T_1, T_2, T_3, … for triangular numbers and S_2, S_2, S_3, … for square numbers?

   ____

2. This arrangement of numbers is often called Pascal's triangle: each new entry is formed from the sum of the two numbers above it.

   (a) Continue the pattern for several more rows.

   (b) Where are the triangular numbers?

   ____

   1
   1 1
   1 2 1
   1 3 3 1
   1 4 6 4 1
   1 5 10 10 5 1

   (c) Where do you find the sum of all the preceding triangular numbers?

   ____

Problem-Solving Objective

To use logical reasoning and number sense to solve problems

Materials

Counters, base 10 materials

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Algebra 2.1
- Data Analysis and Probability 4.1

Focus

This page explores problems based on a conceptual understanding of whole numbers and fractions and uses logical reasoning to consider the problem's context. Backtracking from the final position will help students understand and solve the problems, but counters can also be used to keep track of what is happening. Using a diagram or calculator is another way to sort through the information while keeping the intent of the problem in mind.

Discussion

Page 49 – Market Days

These problems highlight the need to carefully analyze the problem before starting on a solution. For the first problem, Lindsay is clearly not actually going to sell half an egg! At each step, there must be an odd number of eggs. When he sells these eggs, they will both get half of the nearest even number and the customer will get 1 extra egg. This means that when Lindsay had 9 eggs, the last customer must have taken 10. Logic dictates that Lindsay would have had 19 eggs to sell to his third customer. The customer before that must have taken 20 eggs when Lindsay had 39 eggs to sell. So the first customer took 40 eggs when Lindsay had 79 eggs to start with. Counters can be used to work through this thinking, and a diagram can help keep track of what is needed:

Lindsay	third customer	Lindsay	second customer	Lindsay	first customer	Lindsay
9	10					
		19	20			
				39	40	
						79

A diagram can also be used to determine the relationships among the information in the second problem. Base 10 materials could also be used to model the 170 eggs and how they must be distributed over the markets so that, at the second market, the number is 10 less than half of the first.

The third problem is similar to the first problem, but this time the seller must add 2 to the number of chickens left to find half of the number he had to sell. Counters or a diagram could readily show this.

The last problem can also be solved using counters or numbers in a table. At first it seems that there must be 4 or 8 children. When these prove impossible, other ways of finding whole numbers of loaves must be considered while bearing in mind that the number of people and the number of loaves must both equal 12.

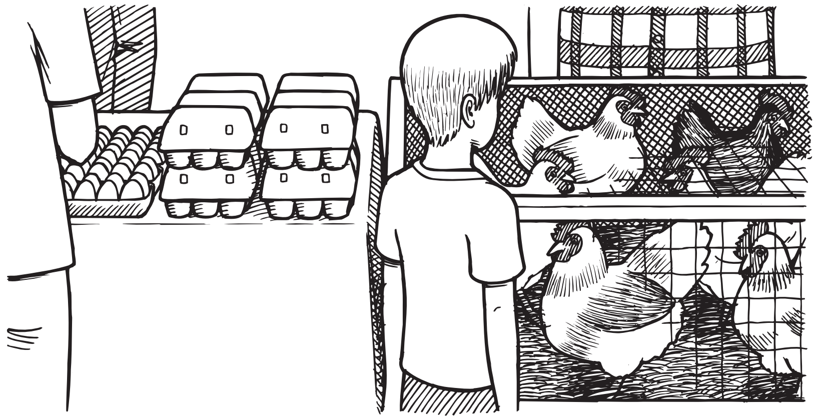
Possible Difficulties

- Unable to see how the half egg, 10 more eggs, or 2 extra chickens fit into the problems
- Not thinking that a full loaf could be made up of 2 halves, 1 half and 2 quarters, or 4 quarters to see how the 12 full loaves could be bought
- Simply working on the basis of calculating the numbers in the problem to obtain incorrect answers

Extension

- Ask for the number of eggs or chickens each customer bought.
- Change the numbers in the problems, but leave the problem statements the same:
  - more eggs at the end of the first problem (it must be an odd number)
  - fewer eggs at the second market (55, 85, 115, … eggs altogether)
  - more chickens available at the end of the day
  - the fourth problem is a very famous problem and would be difficult to change!
- Change the problem's context but leave the numbers the same.
- Have students make up similar problems and challenge others to solve them using diagrams or materials.

8.1 MARKET DAYS



1. Lindsay raises chickens and sells any spare eggs at the local farmers' market. He sold half of his eggs and another half an egg to his first customer, half the remaining eggs and another half an egg to his second customer, and half the remainder and another half an egg to his third customer. At the end of the day, Lindsay noticed that he had 9 eggs left. How many eggs did he have to start with?

____

2. Lindsay sold 170 eggs at two different markets. He noticed that the number he sold at the second market was 10 fewer than half the number he sold at the first market. How many eggs did he sell at each market?

____

3. There was a rush to buy live chickens when the chicken farmer opened his market stall. The first customer bought half of the chickens he had brought to the market and two more chickens. His next customer bought half of the remaining chickens and another two chickens. The third customer bought half of what he had left and another two chickens. Sadly, the fourth customer was only able to buy one chicken. How many chickens did the chicken farmer bring to the market to sell?

____

4. After the market, Lindsay joined a group of people going on a picnic. The 12 of them went first to the bakery and bought a dozen loaves of bread. Each man bought two loaves, each woman bought half a loaf, and each child a quarter of a loaf. How many men, women, and children were at the picnic?

____

Problem-Solving Objective

To use logical reasoning and number sense to solve problems

Materials

Counters

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Algebra 2.1
- Data Analysis and Probability 4.1

Focus

This page explores problems based on a conceptual understanding of whole numbers and fractions and an understanding of what makes sense in the problem contexts. Backtracking from the final position will assist in understanding and solving the problems, but counters can also be used to keep track of what is happening. Using a diagram or calculator are other ways to sort through the information while keeping the intent of the problem in mind.

Discussion

Page 51 – The Farmers Market

These problems highlight the need to carefully analyze the problem before starting on a solution. In the first problem, more than half the turnips are sold on Saturday if 25 fewer than half that number are sold on Sunday. One way to solve the problem involves "try and adjust," using a table to keep track:

	Saturday	Sunday	Total Sold
Try	300	125	425 – too few
Try	400	175	575 – too many
Try	390	170	560 – too few
Try	398	174	572

A form of algebraic reasoning is another way:

The number sold on Sunday is (half the number sold on Saturday – 25).

The number sold, 572, is the number sold on Saturday + (half the number sold on Saturday – 25).

1.5 × the number sold on Saturday – 25 is 572, so 3 × the number sold on Saturday – 50 is 1,144.

3 × the number sold on Saturday is 1,194.

398 turnips are sold on Saturday.

In the second problem, clearly Lance is not actually going to sell half a tomato! At each step, there must be an odd number of tomatoes. When he sells these tomatoes, they will both get half of the nearest even number and the customer will get 1 extra tomato. This means that before Lance sold the last tomatoes, the third customer must have taken 56, leaving Lance 55 tomatoes to sell. The customer before that must have taken 112 tomatoes, leaving Lance 111 tomatoes to sell. So the first customer took 224 tomatoes, leaving Lance 223 tomatoes to sell. Lance started with 447 tomatoes:

Lance	Third Customer	Lance	Second Customer	Lance	First Customer	Lance
55	56					
		111	112			
				223	224	
						447

The third problem is similar to the second problem, but this time the seller must add 3 to the number of pumpkins left to find half of the number he had to sell. Counters, a diagram similar to Problem 2, or a "try and adjust" table show this readily.

Try	Lance	First	Lance	Second	Lance	Third	Lance/last
50	22	28	8	14	1	7	too few
60	27	33	X				
58	26	32	10	16	2	8	too few
62	28	34	11	17	X		
64	29	35	X				
66	30	36	12	18	3	9	3

He brought 66 pumpkins to the market.

Possible Difficulties

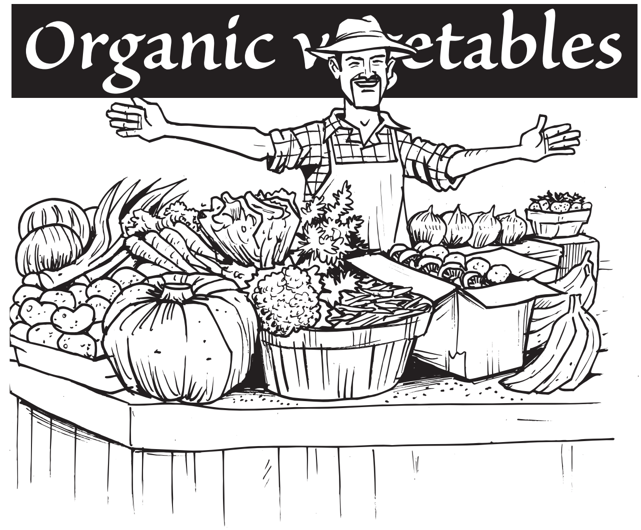
- Unable to see how the half tomato or pumpkin or extra tomatoes or pumpkins fit into the problems
- Simply making calculations with the numbers in the problem to obtain incorrect answers

Extension

- Have students make up similar problems of their own and challenge others to solve them using diagrams, "try and adjust" tables or algebraic thinking.

9.1 THE FARMERS MARKET

1. Lance grows and sells organic vegetables at the local farmers market. He sold 572 turnips at the two markets held last weekend. When he took stock on Sunday night, he noticed that the number of turnips he sold on Sunday was 25 fewer than half the number he sold at the Saturday market. How many turnips did he sell at each market?



____

____

____

2. Lance is well known for his delicious tomatoes. On Saturday, he sold half of his tomatoes and another half a tomato to his first customer, half the remaining tomatoes and another half a tomato to his second customer, and half the remainder and another half a tomato to his third customer. He then sold the remaining 55 tomatoes to the woman who ran the sandwich stall. How many tomatoes did he have to start with?

____

____

3. Since the weather had turned cold and people wanted to make soup, there was a rush on pumpkins as soon as Lance opened his market stall. The first customer bought half the pumpkins he had brought to the market and three more pumpkins. His next customer bought half of the remaining pumpkins and another three pumpkins. The third customer bought half of what he had left and another three pumpkins. The fourth customer quickly bought the last three pumpkins. How many pumpkins did he bring to the market to sell? (Lance sold only whole pumpkins, since he found it too hard to cut them in half—he always got two unequal parts.)

____

____

Problem-Solving Objective

To use strategic thinking to solve problems

Materials

Grid paper, counters in several different colors

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Algebra 2.1
- Data Analysis and Probability 4.1

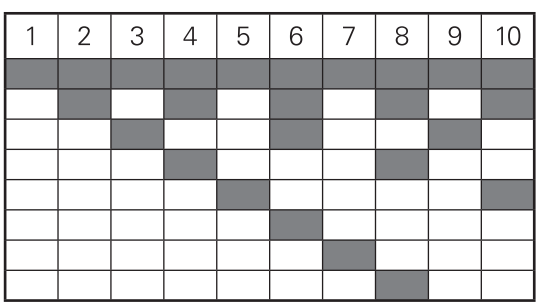
Focus

These pages explore more complex problems in which the most difficult step is to determine what the question is asking. Using materials to help is one way this can be done. Another is to use a diagram to assist in thinking backwards or making trials and adjusting to find a solution that matches all of the conditions.

Discussion

Page 53 – Abstract Art

The first problem can be solved by using colored counters on a grid or coloring the squares to see what is happening. The number of possibilities can then be seen directly or patterns can be sought:



Analysis of the patterns shows that the squares represent the factors of the number of the column in which they occur. Determining the factors of each number from 1 to 50 shows that 48 has the most factors. For the fifth problem, looking at multiples of 48 will help students find the number from 1 to 150 with the most factors.

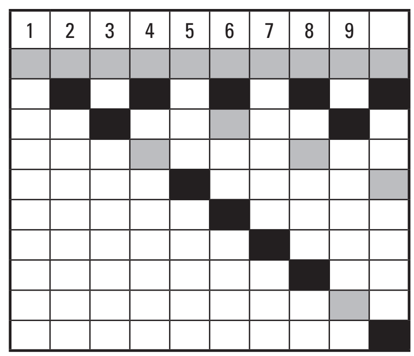
Page 54 – Time Taken

In the first problem, consider the distances covered each hour (a table or list would help). A difference of 25 miles is needed to allow for the one hour of travel and one hour of rest. After 5 hours, cycling at 10 miles per hour covers 50 miles and cycling at 15 miles per hour covers 75 miles. Traveling one more hour at 10 miles per hour would give 60 miles, while traveling one less hour at 15 miles per hour would also give 60 miles. He would need to travel for 5 hours at 12 miles per hour.

The flashing light problem can be solved by considering multiples of 2, 7, and 5. A diagram will help show what is happening in the third problem.

Page 55 – Changing Lockers

These questions are solved by considering prime numbers and squares of prime numbers and systematically examining the pairs of factors for each number. Squares can be colored or counters placed on a grid to show when a door is open:



A door is changed an odd number of times to remain open. These locker numbers are square numbers (which have one number as a factor twice, so an odd number of factors): 1, 4, 9, 16, 25, 49, ... A prime number has only two factors, and these lockers will end up closed.

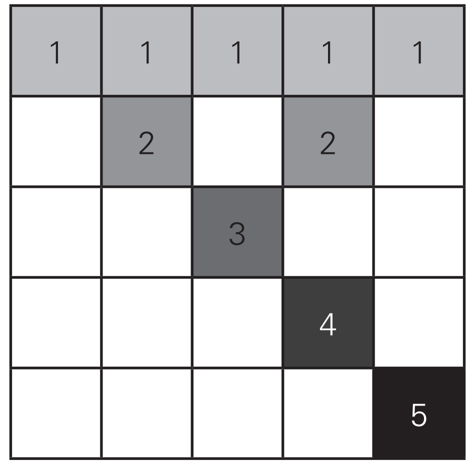
Possible Difficulties

- Not using a diagram or table to come to terms with the problem conditions
- Unable to see how to connect the time cycled to the distance traveled

Extension

- Use different speeds and times for the problems on page 54.

10.1 ABSTRACT ART



Justin made a grid with 50 rows and 50 columns. He asked his friends to help him color squares on the grid to make an abstract design for his art class. He started and colored all of the squares in the first row.

His first friend colored every second square on the second row. The next friend colored every third square on the third row and so on, until every row had some squares colored.

1. When the design is complete, which column would have the most squares colored?

____

Some columns would only have 2 squares colored.

2. Which columns would they be? ________________________

Other columns would have only 3 squares colored.

3. Which columns would they be? ________________________

4. (a) Would any other column(s) have an odd number of squares colored?

   ____

   (b) Why do most columns have an even number of squares colored?

   ____

Justin's teacher was very impressed with his design and decided to make a similar project for the whole class. On the wall of the art classroom, he drew a grid that had 150 rows and 150 columns. The students then took turns coming to the wall and coloring the squares following the same pattern.

5. Which column(s) would have the greatest number of squares colored in the class project?

____

____

10.2 TIME TAKEN

1. A cyclist must meet his friend for coffee at noon. He knows that if he travels at 15 miles per hour, he will arrive an hour early, and if his speed is 10 miles per hour, he will be an hour late. At what speed should he travel in order to arrive on time?

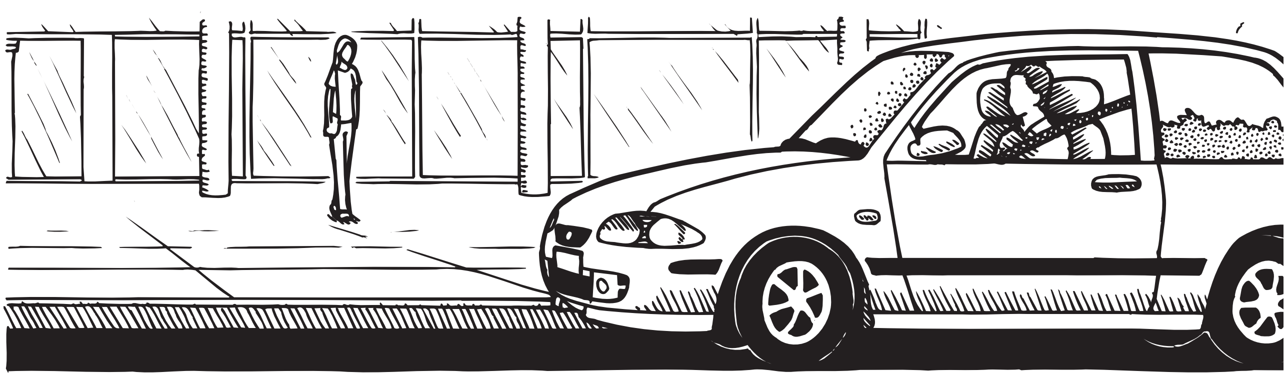
____

2. At the entrance to the town, three colored lights were installed to draw motorists' attention to the town's 30 miles-per-hour nighttime speed limit. The blue light flashes every 2 minutes, the red light flashes every 3.5 minutes, and the green light flashes every 5 minutes. When the lights come on at dusk, they all flash together. How long will it be before they next flash together?

____

3. When Ernest got his driver's license, he agreed to collect his sister, Anna, from the train station on her way home from work. He usually left home at the same time every day and met her at the station at 6:30 p.m., but today she got an earlier train and arrived an hour earlier. Anna started walking home until Ernest saw her and was able to drive her the rest of the way home. Assume Ernest drives at a constant speed and Anna walks at a constant speed. If they arrived home 24 minutes earlier than usual, how long had Anna been walking before Ernest picked her up?

____



10.3 CHANGING LOCKERS

The local high school has exactly 1,000 students, each of whom has a locker. The lockers are along the hallways and numbered 1 to 1,000. To raise money for local charities, the students organized a competition with an entry fee of $1.00.

All 1,000 students had to run past, opening or shutting locker doors:

- the first student opened the door of every locker
- the second student closed every locker door with an even number
- the third student changed every third locker, closing those that were open and opening those that were closed
- the fourth student changed every fourth locker, and so on

1. The student or students who could predict ahead of time which lockers would be open would choose the charity that would receive the $1,000.00.

   (a) What would you predict?

   ____

   ____

   (b) Would any lockers remain open after 10 students had passed along the rows of lockers?

   ____

   ____

   (c) Which lockers would be open after 50 students had changed 50 lockers?

   ____

   ____

   (d) Can you see a pattern for the numbers of the open lockers?

   ____

   ____

   (e) Use your pattern to figure out which lockers would be open after all 1,000 students had run past.

   ____

   ____

   (f) What are the numbers of the first six lockers that changed only twice?

   ____

   ____

   (g) Can you see a pattern for the numbers on these lockers?

   ____

   ____

Problem-Solving Objective

To organize data and use number understanding to solve problems

Materials

Calculator, 0–99 chart

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Data Analysis and Probability 4.1

Focus

These pages explore problems that rely on an ability to carefully analyze the relationships among the data and use an understanding of numbers to keep track of the possible answers. Putting the various interrelated pieces of information into a table is a very helpful way of approaching the problems and provides a systematic way of dealing with a range of problems that have several overlapping conditions.

Discussion

Pages 57 and 58 –School Records/The Town's History

These pages should be used in conjunction with each other. The problems require a clear understanding on the part of students about how pages in a book are numbered. Each side of a sheet of paper that makes up a book is one page. This may lead some students to divide the total number of digits that Kellie wrote by 2. This could not be correct, since it implies that each page would be numbered with a one-digit number, and there are clearly more than 9 pages. There will be pages on which she pencils two-digit numbers and others where she writes three-digit numbers.

Number of digits on a page	Number of pages	Number of digits Kellie penciled
1	9	9
2	90	180
3	166	498

Kellie writes 189 digits for the one- and two-digit numbers. Since 687 digits were written altogether, 498 digits were written on three-digit pages. There were 166 three-digit pages, and Kellie had numbered 265 pages.

A 0–99 chart will help students to organize and keep track of the number of times a certain digit is written—for example, 1:

1 and 2 digit numbers:	1 is written 20 times
100–199:	1 is written 120 times
200–265:	1 is written 17 times

Halfway through this task, she had numbered page 133 and written 291 digits. The problems on page 58 extend this thinking to four-digit numbers.

Page 59 – Team Photos

These problems require a knowledge of division (including remainders, multiples, and factors). There are many ways they can be solved—using counters to see what is happening, try and adjust using different numbers and placing these attempts in a table to organize an approach, and using some form of simple algebraic thinking. Analyzing Problem 1 shows that to find an answer, (7 × number of rows) + 4 must equal (8 × number of rows) – 3:

Number of rows	(7 × number of rows) + 4	(8 × number of rows) – 3
4	32	29
5	39	37
6	46	45
7	53	53

For Problem 3, students must use a similar type of algebraic thinking, multiplying each of the possible number of pages by the two variations of photos (5 + 2 extra, and 8 + 3 extra). For example, (3 pages × 5) + 2 extra = 17. As with Problem 1, students must continue until they find a number of photos that matches both (number of pages × 5) + 2 and (number of pages × 8) + 3.

Possible Difficulties

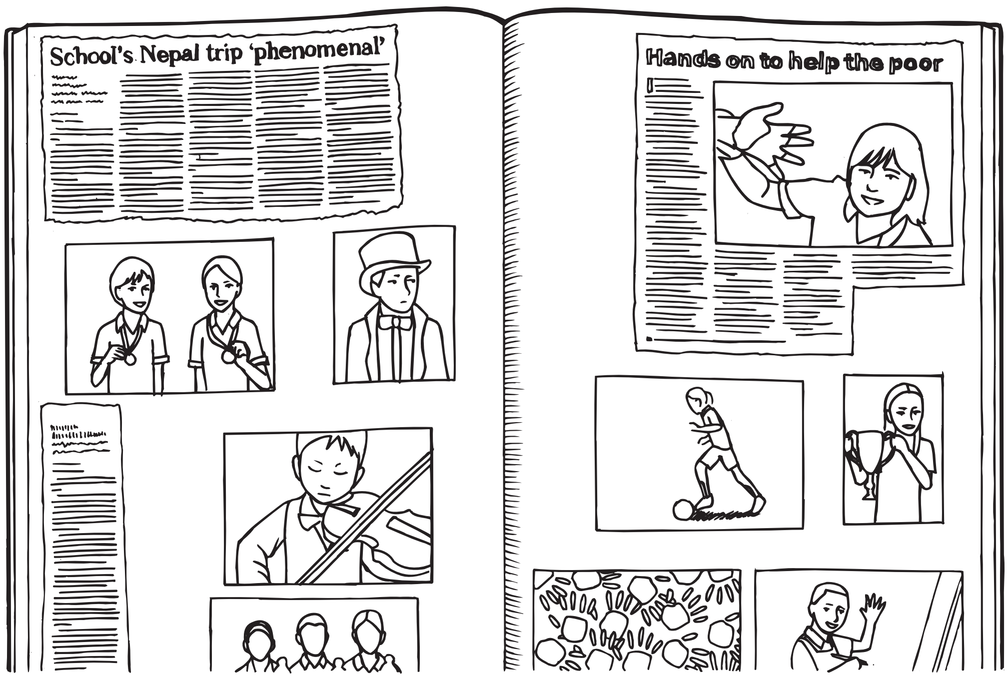
- Confusing the page numbers with the number of digits to provide an answer of 687 pages
- Not taking into consideration the remainders when considering multiples

Extension

- Extend these problems to other situations where pages are numbered.
- Provide other combinations of photos and pages for the team book problems.

11.1 SCHOOL RECORDS

Kellie's school celebrated its 50th anniversary this year. She joined with a group of her classmates to gather all the photographs and newspaper articles they could get from former students and put together a record book of life at the school over the last 50 years. It was interesting to see how much life had changed over the years.



When they had all of the materials arranged on the pages of the school's collection of its records, Kellie penciled the page number on each page to help later with drawing up the table of contents.

1. After she had finished, Kellie told the others that she had written 687 digits. How many pages had she numbered?

    ____

2. How many times had she written the digit 1? ____________________

3. What page was she numbering when she was halfway through the book?

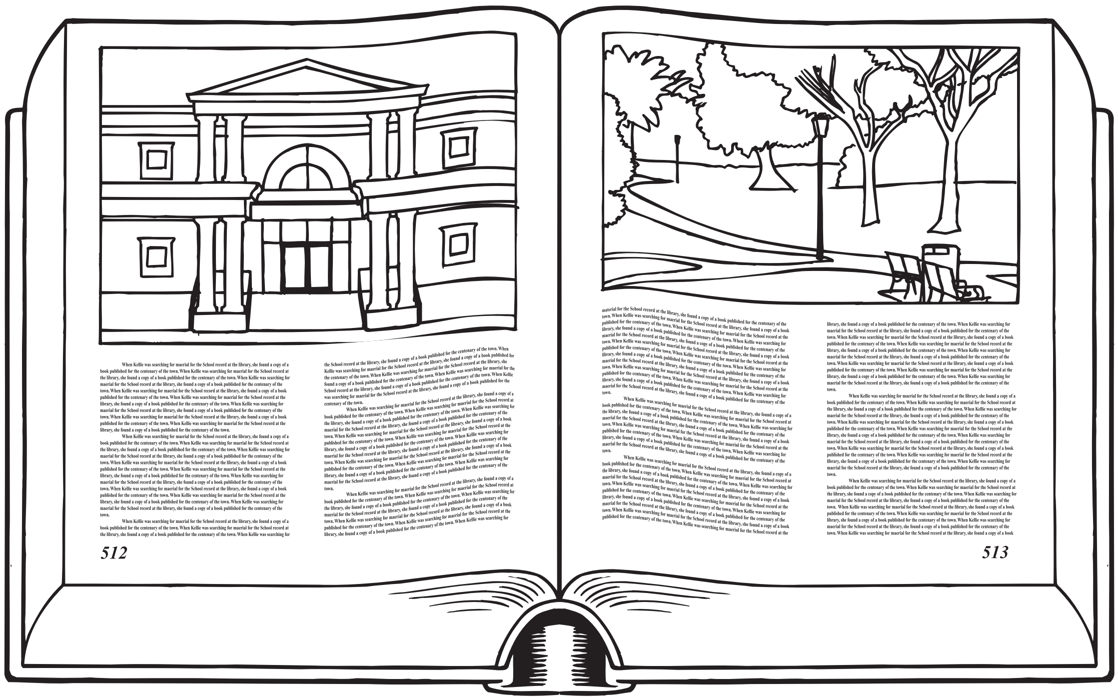
    ____

4. How many digits had she written when she finished numbering that page?

    ____

11.2 THE TOWN'S HISTORY

While Kellie was searching at the library for material for the school's records, she found a copy of a book published for the 100th anniversary of the town.



1. Afterwards, Kellie said to her friends, "I'm glad I didn't have to write the page numbers on the pages of that large book." If there were 1,027 pages in the book, how many digits would Kellie have had to write?

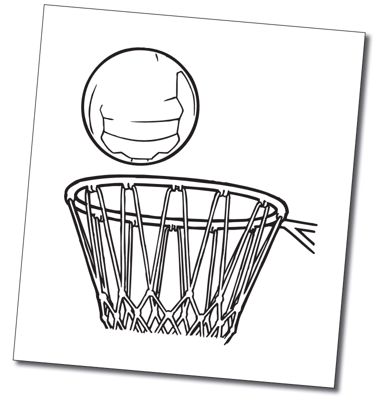
2. Her friend Bob has always said that his lucky number is seven. How many times would Kellie have written the digit 7?

3. What digit would Kellie have written most often?

4. When Kellie looked at the book, she noticed that it had been typed rather than produced on a word processor. She then said to Bob, "Imagine if you had to type the page numbers for the last 500 pages of the book!" How many digits would Bob have typed to number those pages?

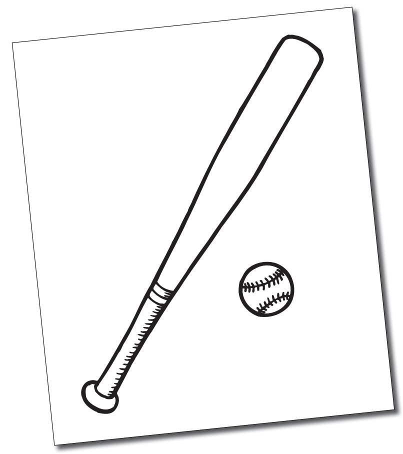
11.3 TEAM PHOTOS

1. Kristen is a keen basketball player. She notices that when the players on her team are being organized on the bleachers for the end-of-season photo, if 7 players are seated in each row, 4 players do not get a seat. When they move closer together to seat 8 players in each row, they have enough seats for the coach, the assistant coach, and the first-aid person. How many ball players are there in Kristen's club?



____

2. Kyle is uploading photos of his baseball team onto his website. He knows that most of his friends would not be interested in the site if he only put one photo on each webpage, so he decides to scale down each photo so the team can see several photos on a page at the same time. After he has placed 8 photos per page, he still has 4 photos left. When he puts 12 photos on each page, he needs 3 fewer pages. How many photos does he have to upload, and how many webpages does he need if he puts 12 photos on each page?



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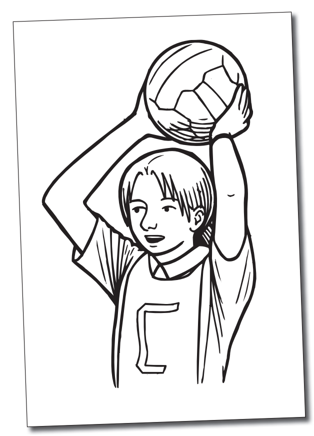
3. Kristen decides to upload the photos of the players on her team, taken at last season's games, onto a website like Kyle's. Some photos show a lot of action and need to be quite large, while others are of only one or two players and could be smaller. At first, she decides to put 5 photos on each webpage, but this takes up a lot of pages, and the last page has only two small photos. When she changes the arrangement to 8 photos on a page, the last page has 3 larger photos. She has more than 45 photos. How many photos does Kristen have?



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4. Kristen wants to have the same number of photos on each webpage. How many pages would she need to make to do this?

____

Problem-Solving Objective

To use spatial visualization, logical and proportional reasoning, and an ability to rename among fractions and percents to solve problems

Materials

Counters, calculator

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Data Analysis and Probability 4.1

Focus

This page explores different ways of visualizing the problem situation and analyzing the possibilities that make up the whole solution. Logical reasoning is needed, as well as an understanding of the measurement concept of direction. Materials, diagrams, or tables can be used to organize, sort, and explore the data.

Discussion

Page 61 – After Work

Using a table to keep track of the times taken in the first problems will enable the different pieces of information to be brought to bear:

Time	5:00	5:01	5:02	5:03	5:04	5:05	5:06
Jared	6	5	5	5	5	4	3
Cath	6		5		4	4	4

5:07	5:08	5:09	5:10	5:11	5:12	5:13	5:14
3	3	3	2	2	2	2	1
4		3		2		1	

The other elevator problems can be solved in the same way: leaving from floor 8 they do not arrive at the same time —the fast elevator arrives first. From floor 9, the slow elevator is quicker—5:22 compared to 5:23.

The problem with the passengers on the bus can be solved by “try and adjust” (the numbers in the problem suggest that the number of passengers must be a multiple of 5, 2, and 3) or by using the fractions to work backwards. If counters are used to represent the passengers and then adjusted to reflect the fractions involved, it is much easier to see what to do when working backwards.

Possible Difficulties

- Immediately thinking that the slow elevator will get there last
- Unable to construct tables or draw diagrams to show the relationships among floors

Extension

- Change the speed of the elevator, the time spent waiting for passengers, and the floor numbers.
- Groups of students could write their own problems involving percents and fractions and challenge others to solve them.

12.1 AFTER WORK

1. Cath and Jared work on the 6th floor of their office building. When they leave the office to go home at 5:00 p.m., Cath gets in the slow elevator, and Jared gets in the fast elevator. The fast elevator takes 1 minute between floors, and the slow elevator takes 2 minutes between floors. The first elevator to reach a floor stops for 3 minutes to pick up other office workers on their way home. At that time of day, every floor has workers waiting to get on.

   (a) Who will be the first to get to the lobby on the first floor?

   ____

   (b) How long will it take for each of them?

   ____

   ____

   Their friends, Kate and Jon, work in an office two floors above them. They believe it makes no difference what elevator they get in.

   (c) Are they correct?

   ____

   ____

   (d) What if they worked on the 9th floor?

   ____

   ____

2. When Cath and Jared left their office building, they caught a bus to a restaurant downtown. They noticed an unusual pattern in the way the passengers got on and off the bus:

   - At the first stop, $\frac{2}{5}$ of the people got off and $\frac{3}{5}$ of the original number got on.
   - At the second stop, $\frac{1}{2}$ of the people got off and $\frac{1}{3}$ of the number that were left on the bus got on.
   - When they got off at the next stop, $\frac{3}{4}$ of the people got off and there were only 5 people to continue on their journey.

   How many people were on the bus before these three stops?

   ____

   ____

Problem-Solving Objective
To use strategic thinking to solve problems

Materials
Calculator

NCTM Content Standards
- Number and Operations 1.1, 1.2, 1.3
- Data Analysis and Probability 4.1

Focus
This page explores problems that may have several answers, and further analysis of the data is needed to determine if this is the case or if there is a unique solution. Knowledge of simple percents is also required. A process of "try and adjust" can be used; however, reasoning logically about the possibilities and using a table or diagram to organize them will be more productive. These ways of thinking can then be generalized to other complex problems.

Discussion
Page 63 – The Fish Market

There are several ways these problems can be solved. For Problem 1, the "try and adjust" approach can be used: Fix the cost of the fish and shrimp at $64 and the cost of the shrimp and oysters at $33. Try $40 for fish and $24 for shrimp. At this price the oysters must be $9. This gives too small a cost for the fish and oysters ($49 instead of $60). Using a table makes it easier to try various prices:

	Fish	Shrimp	Oysters	Fish & Shrimp	Shrimp & Oysters	Fish & Oysters
Try	40	24	9	64	33	$49 – too little
Try	45	19	14	64	33	$59 – too little
Try	46	18	15	64	33	$61 – too much
Try	45.50	18.50	14.50	64	33	$60

A form of algebraic reasoning can also be used. Since the fish and oysters cost $60 and the shrimp and oysters cost $33, the difference in price between the fish and the shrimp must be $27. Since the fish and shrimp cost $64, two fish cost $27 + $64 or $91. A fish costs $45.50, the shrimp cost $18.50, and the oysters cost $14.50.

The second problem is made simpler to solve by changing metric tons to kilograms. Altogether, 3,200 kg + 4,800 kg + 4,320 kg of shrimp are caught and sold. If shrimp are sold seven days a week, a total of 12,600 kg are needed to meet demand. Instead, only 12,320 kg were caught, leaving a shortfall of 280 kg. There are enough shrimp if they are sold 6 days a week.

The third problem can also be solved by the "try and adjust" approach using a table and starting with the value of $60 for the salmon (since it appears to be more expensive). Using algebraic reasoning is more direct. Subtracting the total prices shows that the difference in price for the two types of fish is $12. Adding the total prices shows that the total cost of 5 kg of each type of fish is $300, so the total cost of the two types of fish is $60. Two kilograms of salmon must cost $72, so the salmon costs $36 per kg and the cod costs $24.

The fourth problem is similar to the first: squid costs $5.50 per lb and mussels cost $4.75 per lb.

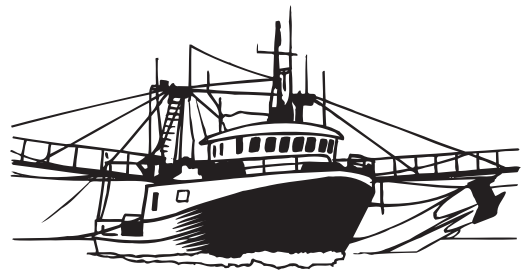
Possible Difficulties
- Not using a table or diagram to manage the data when taking the "try and adjust" approach
- Keeping only one condition in mind when there are two aspects to be considered
- Unable to see how to change the given prices to find sums and differences in order to get the price of two of an item

Extension:
- Discuss the various methods used by students to solve the problems. Include the examples already discussed. Ask them to solve each problem using a different method from the one they tried first.
- Encourage students to use algebraic thinking in considering the relationships among the information. Some students may be able to diagram this or use symbols.
- Challenge students to change the numbers in the problems while keeping the same solutions.
- Students could also write problems using different contexts and larger numbers for others in the class to try.

13.1 THE FISH MARKET

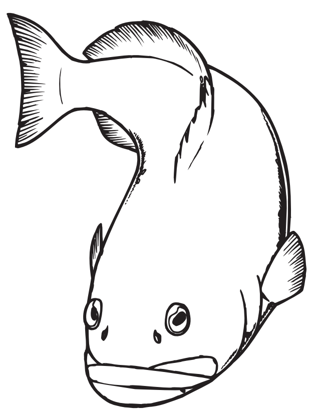
1. Frances bought some fish, some shrimp, and some oysters. When combined, the fish and the shrimp amount to $64, the shrimp and the oysters total $33, and the fish and the oysters cost $60. How much did each of the different types of seafood cost?

    ____



2. Due to bad weather, the shrimp trawler could only go out to catch shrimp 3 days this week. On the first day, the trawler netted 3.2 metric tons of shrimp. It then brought in 1.5 times as much on the second day, and 10% less on the third day than on the second day. Normally, the fish market sells 1,800 kg of shrimp a day. Did the trawler catch enough to satisfy the shrimp buyers this week?

    ____



3. On Wednesday morning, when Frances and her friend Fiona went to the fish market, the fishing boats had brought in lots of fresh fish. Fiona paid $144 for 2 kg of salmon and 3 kg of cod. Frances bought 2 kg of cod and 3 kg of salmon for $156. What was the price per kg for each type of fish?

    ____

4. (a) During the weekend, Fiona bought some frozen seafood at the market: a large tray of sardines, a 3-lb bag of squid, and a 2-lb box of mussels. If the sardines and the squid cost $50, the sardines and the mussels cost $43, and the squid and the mussels cost $26, how much did she pay for each type of seafood?

    ____

    (b) How much do the mussels cost per lb? ______________________________

    (c) How much does 1 lb of squid cost? ______________________________

Problem-Solving Objective

To organize data and use number understanding to solve problems

Materials

Calculator

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Data Analysis and Probability 4.1

Focus

These pages explore problems that call on an ability to carefully analyze the relationships among the data and organize the information gained to keep track of the possibilities. Putting the various interrelated aspects into a table or diagram provides a systematic way of dealing with the overlapping conditions.

Discussion

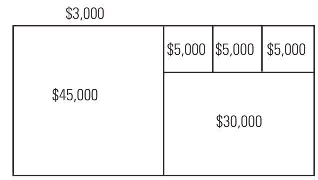
Page 65 – Money Matters
The first problem can be solved by "try and adjust" using a table to keep track of the information:

Amount	$220		$200	$150
	Mother	Son	Mother + daughter	Son + daughter
Try	$120	$100	$80	$180 – too much
Try	$130	$90	$70	$160 – too much
Try	$140	$80	$60	$180 – too little
Try	$135	$85	$65	$150 – correct

A form of algebraic reasoning can also be used: Subtracting the amount the son and daughter spent from the amount the mother and daughter spent shows that the difference between what the mother and son spent is $50. Adding this to the amount the mother and son spent shows that 2 times what the mother spent is $270. The mother spent $135, the son spent $85, and the daughter spent $65.

Working backwards solves the second problem, but the total distributed has to be kept in mind to work out the shares. Desney receives $90, which is two-thirds what was left after Carol received $9 and one-third, so Carol got $54. At this point, $144 has been distributed, and this is two-thirds what was left after Brian received $4 and one-third, so Brian got $76 and Aunt Alice gave them $220.

The third problem is best solved using a diagram to see that after the $3,000 is removed, one-half or $45,000 goes to the daughter, two-thirds of what remains or $30,000 goes to the grandson, and the remainder of $15,000 is split equally among the three great-grandchildren, who get $5,000 each.



Page 66 – Scoring Points
The first three problems on the page can be solved in a similar manner by drawing up tables to see how the points are allocated and choosing the results that match all conditions. For the last problem, adding the attendances gives two times the number who went to the first three sessions. Session 4 must have had (118 – half of 194), or 21 attendees.

Page 67 – Puzzle Scrolls 1
The puzzle scrolls contain a number of different problems, all involving strategic thinking to find possible solutions. In many cases students will find that tables, lists, and diagrams are needed to manage the data while exploring the different possibilities. In Problem 3, the "smallest number possible" has been added to limit the solutions.

Possible Difficulties

- Trying to manage the data without using a table or diagram
- Not considering all aspects and information in the puzzle scrolls

Extension

- Challenge students to figure out how many people went to each of recording sessions 1, 2, and 3 of the televised quiz show.
- Change the numbers and scenarios to write other problems based on the puzzle scrolls.

14.1 MONEY MATTERS

1. A mother went holiday shopping with her son and daughter. The mother and son spent a total of $220, the son and daughter spent a total of $150, and the mother and daughter spent $200 total.

   How much did each of them spend?

   ____

   ____

2. Aunt Alice promised her nephew Brian and nieces Carol and Desney some money to divide up to help them with their shopping expenses. Before going shopping, they first had to figure out the share each would get from the way their aunt planned to distribute the money:

   - Brian was to get $4.00 for a milk shake and one-third of what was left
   - Carol was to get $9.00 for pizza and salad for her and Desney and then one-third of what was left.
   - Desney would get the remaining $90.00.

   (a) How much money would Brian and Carol get?

   ____

   (b) How much money did their aunt give them altogether?

   ____

3. Great-Grandma Jean left $93,000 in her will. She asked that it be divided so that each of her three great-grandchildren would receive the same amount, their father (her grandson) twice as much as the three great-grandchildren together, and her daughter (the children's grandmother) $3,000 more than the father and great-grandchildren together. How much does each get?

   ____

   ____

14.2 SCORING POINTS

In a television quiz show, each contestant is given 10 questions to answer. Five points are won when a question is answered correctly, and 3 points are lost for a question that is not answered correctly or not answered at all. If a contestant's score drops below zero, he or she must leave the program immediately.

1. How many questions must be answered correctly to score 34 points?

____

2. What are all the possible scores for the quiz at the end of the program?

____

3. Because there weren't many different scores for the people in the quiz show, the rules were altered so that 3 points were given for a correct answer, 1 point was lost for each incorrect answer, and no points were given or lost for a question that was not answered at all. If there were 20 questions and a contestant scored 48, how many answers could have been correct and how many not answered?

____

4. A new quiz show debuted with a more attractive format. Each question that a participant answered correctly led to a prize of $250. A question that was not answered drew a penalty of $500. At the end of the competition, one contestant had answered correctly 7 times as many questions as she answered incorrectly and received $3,750. How many questions were not answered correctly?

____

5. A total of 118 people attended recording sessions for the first four shows. A total of 70 people attended the first two shows, a total of 65 were present at the first and third shows, and a total of 59 people came to the second and third shows. How many people attended the fourth show?

____

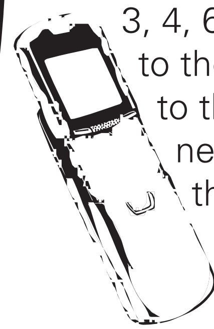
14.3 PUZZLE SCROLLS 1

1. Jill and Ryan race to the top of 80 stairs. Jill gives Ryan a 15-step start. Ryan covers 2 steps per second and Jill 3 steps per second. Who will reach the top first?

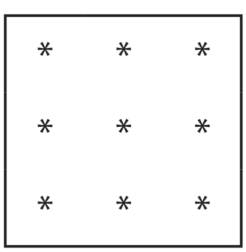
2. Two cars are driving around a 2 km track. One car makes a lap every 70 seconds and the other car every 60 seconds. How long will it take the faster car to be one lap ahead?

3. A farmer counted his cows. If he counted by 7, there were none left over, but with 2, 3, 4, 5, or 6, there was always one left over. What is the smallest possible number of cows in the herd?

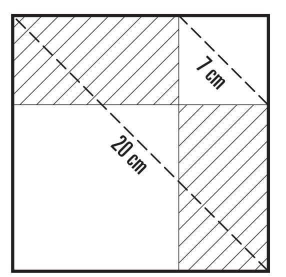
4. The serial number of my phone is a 4-digit number less than 5,000 and uses the digits 3, 4, 6, and 9. The 4 is next to the 9, the 3 is not next to the 4, and the 6 is not next to the 3. What is the number?



5. Draw two squares in the diagram so that all the stars are separated.



6. Find the area of the shaded parts.



Problem-Solving Objective

To use logical reasoning, fractions, and measurement to solve problems

Materials

Calculator

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Algebra 2.1, 2.3
- Geometry 3.2

Focus

These pages investigate relationships among distance and time expressed as fractions of the distance around a running or cycling track. Logical thinking and organization are needed to see how the runners or cyclists progress, keep track of their positions, and determine when they will coincide.

Discussion

Page 69 – Training Runs

Drawing a diagram to show the running track and the relative positions of the runners may help students. In Problem 1, there are 24 markers (one every 10 m) around the 240 m track, and these show how the relative jogging rates of one-half, one-fourth and one-third can be determined. In one minute, Heather jogs 120 m, Hannah jogs 80 m, and Helen jogs 60 m. This information can be organized to keep track of how far each has jogged and when each reaches the starting line after jogging 240 m. One way is to place the information in a table to show how far each girl has jogged after 1, 2, 3, and so on minutes:

Time in Minutes	Heather	Hannah	Helen
1	120	80	60
2	start	160	120
3	120	start	180
4	start	80	start
5	120	160	60

After 4 minutes, all of the girls have reached the start at least once, but not all at the same time.

Other ways to find a solution can also be used; for example, while the table above focuses on the distance traveled, other students may prefer to work with time, using the pattern that Heather reaches the start every 2 minutes, Hannah every 3 minutes, and Helen every 4 minutes. Some students may quickly follow a pattern or seek a common multiple of 2, 3, and 4, which is 12.

The second problem can be solved in a similar manner: Expressing the relative distances run as written fractions may require further thought for some students, and this problem asks for the distance traveled rather than the time taken. Students must calculate the total distances traveled and then convert the meters to kilometers.

Page 70 – Riding to Work

In the first problem, Gary is halfway around the track and cycles 600 m in 1 minute. Jeff is one-third of the way around the track and cycles 400 m in 1 minute. Exploring how far each rider has cycled after 1, 2, 3 … minutes shows that both cyclists are at the start after 6, 12, 18, … minutes. Another way is to see that Gary reaches the start every 2 minutes, Jeff every 3 minutes—6, 12, 18 … are common multiples of 2 and 3. Since they cross the start together every 6 minutes, there will be 7 more times, as well as the time they started, to give 8 times at the start altogether.

Analyzing the second problem shows that Jeff rides $\frac{24}{56}$, or $\frac{3}{7}$, of the distance around the track before they meet. Gary must have cycled $\frac{4}{7}$ of the distance in 24 seconds. He would take 6 seconds to travel $\frac{1}{7}$ of the distance and 42 seconds to complete one lap. Put this information into a table:

Time in Seconds	24	48	72	96	120	164	168
Jeff	$\frac{3}{7}$	$\frac{6}{7}$	$\frac{9}{7}$	$\frac{12}{7}$	$\frac{15}{7}$	$\frac{18}{7}$	$\frac{21}{7}$ (3 laps)
Gary	$\frac{4}{7}$	$\frac{8}{7}$	$\frac{12}{7}$	$\frac{16}{7}$	$\frac{20}{7}$	$\frac{24}{7}$	$\frac{28}{7}$ (4 laps)

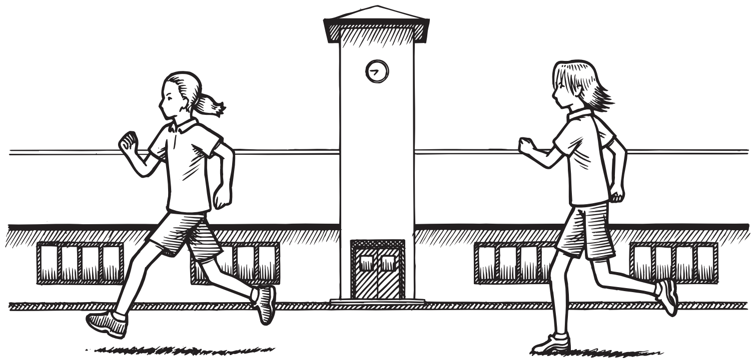
Gary is 1 lap ahead after 2 minutes, 48 seconds. When Gary has completed 60 laps, Jeff has only completed 45 laps and has 15 more to ride. Similar reasoning solves the last problem.

Page 71 – Bike Tracks

The first problem is solved the same way as on the preceding page, only the fractions involved are now $\frac{4}{9}$, $\frac{8}{9}$, $\frac{12}{9}$, … and $\frac{5}{9}$, $\frac{10}{9}$, $\frac{15}{9}$ … . Problems 5–10 can be solved by placing some of the information in a table to see the pattern (98 entries is too many!) or else using the different intervals of 4, 5, 2, and 3 days to get a common multiple of 60. They will only ride together one more time, on a Wednesday. Similar reasoning shows that Georgia and Gina ride together every 20 days.

15.1 TRAINING RUNS

To prepare for the school's cross-country run, Heather and her two sisters, Hannah and Helen, decide to jog on the track each morning. The running track is 240 m around, with markers to show the distance every 10 m and a large clock to keep track of how fast they are going. After one minute, Heather has jogged halfway around the track, Hannah has run one-third of the way, and Helen has jogged one-fourth of the distance around the track.



1. If they all continue jogging at the same rate, after how many minutes will they all cross the starting line at the same time?

____

2. (a) Their cousins, Len, Liam, and Lachlan, also have a cross-country race coming up and decide to follow a similar training schedule. Their running track is 300 m around, with markers to show the distance every 10 m. It also has a large clock. After 1 minute, Len has jogged $\frac{1}{2}$ the track, Liam has jogged $\frac{1}{5}$ of the way, and Lachlan has jogged $\frac{1}{3}$ of the distance. If they all continue jogging at the same rate, how many kilometers will each of the boys have jogged when they first cross the starting line at the same time?

____

____

(b) If they start jogging at 7:30 a.m., what will be the time when they first cross the starting line together?

____

(c) What would be the next time they cross the starting line together, and how far would each boy have jogged by then?

____

____

15.2 RIDING TO WORK

Due to the high cost of gas, Gary and Jeff decided to cycle to work rather than drive their cars. However, on the first week they realized they needed to be a lot fitter to ride fast enough to get through the traffic. They used the 1,200 m cycle track at the local high school for some training. In one minute, Gary rode halfway around the track while Jeff rode one-third of the way.

1. If they both continued to ride at the same speed until they crossed the starting line together, how far would each have cycled?

   ____

2. Heavy rain stopped their training after 45 minutes. How many times would they have been at the starting line together?

   ____

3. To vary their practice, Gary and Jeff went to another cycle track and decided they would start at the same point but ride in opposite directions. Jeff was able to cycle around the track in 56 seconds, and they passed each other every 24 seconds.

   (a) How many seconds did it take Gary to cycle around the track?

   ____

   (b) When would they next meet at the starting point?

   ____

   (c) How many laps of the track would each rider have made at that point?

   ____

4. They then decided that each of them would ride 60 laps. When Gary had finished, how many more laps did Jeff need to cycle?

   ____

5. After some additional training, Jeff found he could cycle one lap in 35 seconds, while Gary still took 42 seconds.

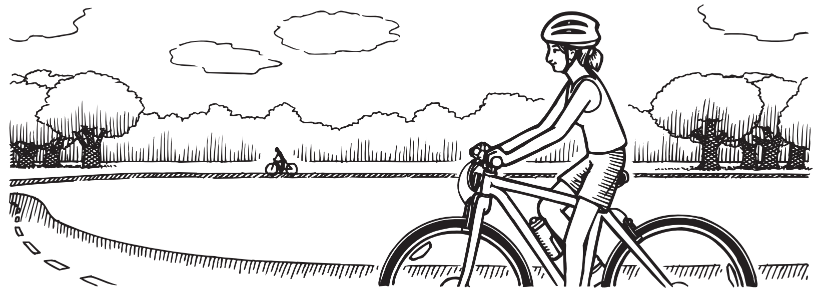
   (a) How long would it take him to be one lap ahead of Gary?

   ____

   (b) How many laps would he be ahead if they continued cycling at these speeds for a total of one hour?

   ____

15.3 BIKE TRACKS



Their friends Georgia and Gina also decided to cycle to work and joined Jeff and Gary in training. They liked the idea of cycling around a track in opposite directions so they could see how each other was doing. When they first started out, Gina cycled around in 72 seconds, and they passed each other every 40 seconds.

1. How many seconds did it take Georgia to cycle around the track?

   ____

2. When would they next meet at the starting point?

   ____

3. How many laps of the track would each rider have made at that point?

   ____

4. They decided that each of them would ride 30 laps. When Georgia had finished, how many more laps did Gina need to cycle?

   ____

During the summer, all four cyclists planned to ride along the bike path by the river after work. Gina decided to ride one night and then take a three-day break. Georgia had four days off between her rides, Jeff cycled every second day, and Gary every third day. On the first Saturday evening of daylight saving, they were all at the track together.

5. When would they next be able to cycle together?

   ____

6. If there were 14 weeks of daylight saving, how many times would they all cycle together?

   ____

7. What days of the week would they be?

   ____

8. When would Georgia and Gina ride together?

   ____

9. When would Gary and Gina ride together?

   ____

10. When would Jeff and Georgia ride together?

   ____

Problem-Solving Objective

To use logical reasoning and an ability to visualize a sequence of events to solve problems using number patterns.

Materials

Calculator

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Algebra 2.1
- Data Analysis and Probability 4.1

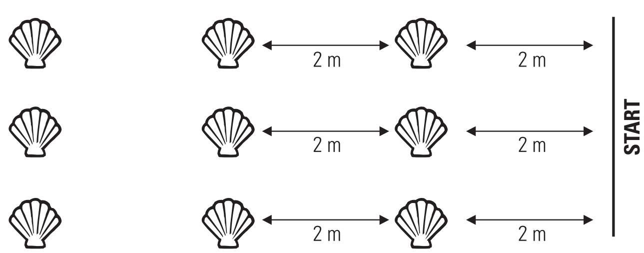
Focus

This page explores problems that involve a large amount of data and require both computation and patterning to determine solutions.

Discussion

Page 73 – Beach Carnival

On this page the information must be carefully analyzed to determine the distance of each shell from the starting line, the number of shells that are set out for the race, and the distance to run to pick up and return each shell to the bucket, on the condition that only one shell can be collected at a time. Some students may use counters to represent the race and see what is happening. Using a diagram is another way:



When the problem is understood, a table that keeps track of the distances to each shell and the cumulative distance covered would enable an answer to be obtained, or even allow students to see a pattern in the results that can be applied to the whole situation.

Shell Number	Distance Run	Total Distance (meters)
1	4	4
2	8	4 + 8
3	12	4 + 8 + 12
4	16	4 + 8 + 12 + 16
5	20	4 + 8 + 12 + 16 + 20

This would suggest that adding all the numbers using a calculator would give the correct answer of 5,100 m, or more than 5 km (which at first seems surprising).

Another way is to look for a pattern:

4, 4 (1 + 2), 4 (1 + 2 + 3), 4 (1 + 2 + 3 + 4), . . .

The total must be 4 × the sum of all the numbers 1–50. This result can be found by using a calculator or a very famous method supposedly demonstrated by mathematician Karl Gauss when he was only a young boy.

Sum wanted is 1 + 2 + 3 + 4 + 5 . . . + 50.

This sum is also 50 + 49 + 48 + . . . + 1.

Twice the sum must be 50 × 51, so the sum is half of 50 × 51. ([50 × 51 = 2,500] ÷ 2 = 1,275)

The winner must run 4 × 1,275 = 5,100 m.

This thinking now has to be applied in reverse to the second problem. Since Jeff picked up 42 shells, the distance he ran must be 4 × the sum to 42 (that is, 4 × half of 42 × 43), which is 3,612 meters; Jeff ran 3 km 612 m.

Possible Difficulties

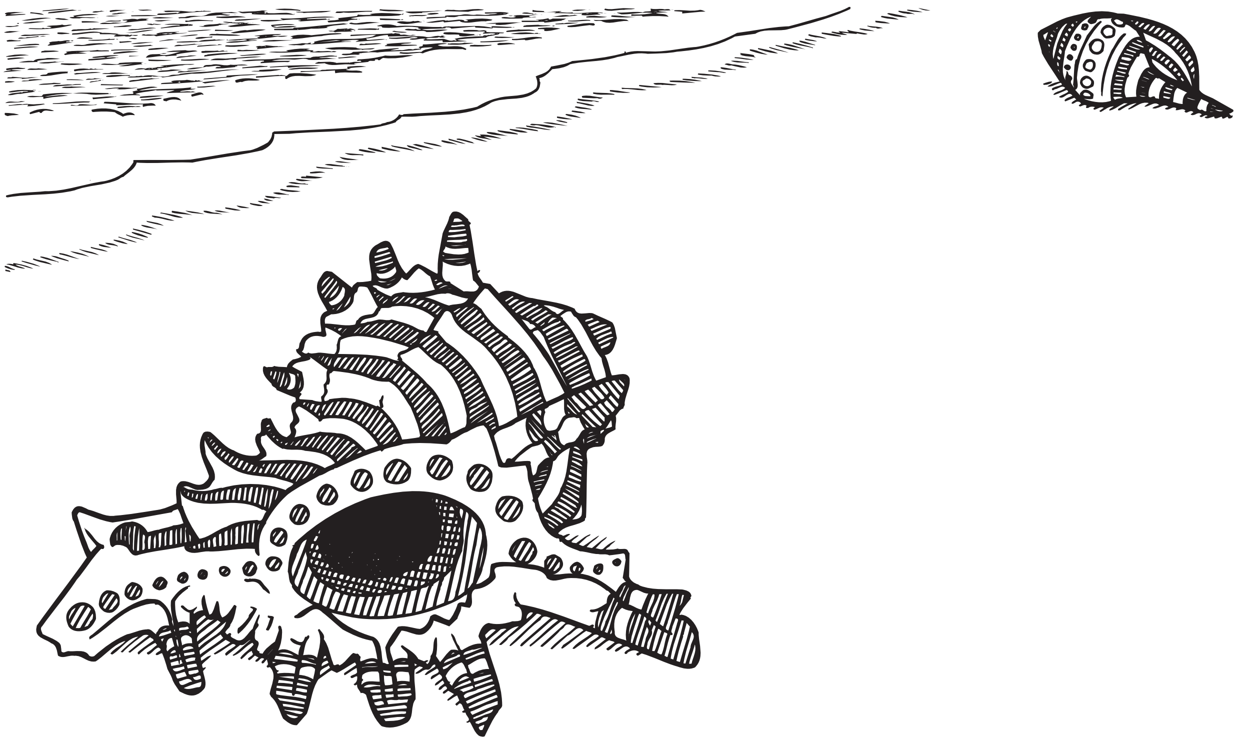
- Not understanding the way in which the shells are distributed from the starting line
- Using only the distances to the shells
- Not attempting a solution because of the complexity of the steps or calculations

Extension

- Use smaller or larger numbers of shells for the race and vary the distance between shells.
- Ask students to research the mathematician Karl Gauss and find out more about his famous solution to the sum of the first 50 numbers.

16.1 BEACH CARNIVAL

At the summer beach carnival the painted shell competition was the most keenly contested. Each of the contestants had a line of shells placed in front of them. In each line, the shells were placed at 2 m intervals. Contestants had to run from a starting line 2 meters before the first shell and collect each shell. Shells could only be collected one at a time, with each shell placed in a bucket at the starting line before running to get the next one.



1. The shells had to be collected in order of first, second, third, and so on until the last shell, 100 m from the starting line, had been collected. The winner was the first person to place all of his or her shells in the bucket.

   How far did the winner have to run?

   ____

2. When the winner had collected all of his or her shells, the competition was finished, and the number of shells in each bucket was counted to see who filled the other places. Jeff had just placed his 42nd shell in his bucket.

   How far had he run?

   ____

Problem-Solving Objecive

To use spatial visualization and logical reasoning to solve problems

Materials

- Cubes in 4 different colors
- Grid paper to draw views of the shapes and nets of cubes
- Digital camera to photograph the shapes

NCTM Content Standards

- Geometry 3.1, 3.2, 3.3, 3.4
- Algebra 2.1

Focus

These pages explore arrangements of three-dimensional shapes to determine how particular outcomes are formed and to investigate the patterns they exhibit. Spatial and logical thinking and organization are required as students investigate all likely arrangements to ensure that the final forms match the given criteria or visualize a given shape in terms of its component parts.

Discussion

Page 75 – Colored Cubes
Students are asked to extend an arrangement of cubes in a Latin square (where each row and column have exactly one each of three or more different-colored cubes) to a cube in which each horizontal and vertical row or column has exactly one cube of each color. Students will need to visualize these patterns to guide their building of the complete cubes, and some degree of "try and adjust" will probably be needed.

Encourage students to create systematic arrangements as they complete the tasks and describe the pattern of the blocks in the middle of the arrangements. This may lead some students to start with the 3-by-3 cube and build layers all around to complete a 4-by-4 cube. All arrangements would be possible with a single cube at the center of an arrangement of an odd number of colors, while arrangements made of an even number of colors would have a 2-by-2 cube in their center. The actual descriptions given will vary according to the arrangement chosen for the initial Latin square, but all are essentially the same with different colors occupying different positions.

Page 76 – Growing Cubes
This problem requires students to visualize how colored cubes are arranged to form larger cubes. Students must bear in mind that a cube with a height and width of 2 cubes will have a total of 8 cubes, a 3-by-3 cube will have 27 cubes, and so on—multiplying side × side × side (or side^3) to find the total. Students can then determine the difference in amount of cubes used; for example, from a single cube to a 2-by-2 cube: $2 \times 2 \times 2$ (or 2^3) – 1 = 7 cubes. The next difference is 27 – 8 = 19, and so on. Other students will see this pattern in terms of the amounts of cubes that are added to build the next cube. Both approaches are satisfactory and will help students see a pattern in terms of how many cubes each subsequent size increases by: sequential multiples of six.

Page 77 – Viewing Cubes
This set of investigations encourages students to see three-dimensional shapes in terms of the component two-dimensional forms. Viewing from above, the front and one side is sufficient to capture the details of a three-dimensional shape. Another way of providing a plan for a building is to use a grid that tracks the number of cubes in each position of the base. After students make a shape of their own and construct a grid plan for another student to follow, they may take digital photos from the side and front to compare with the original.

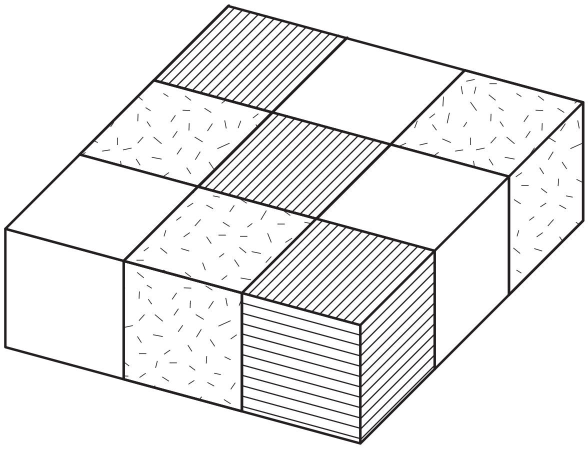
Possible Difficulties

- Unable to visualize the shapes from the two-dimensional representations
- Unable to keep track of the possibilities in any of the investigations

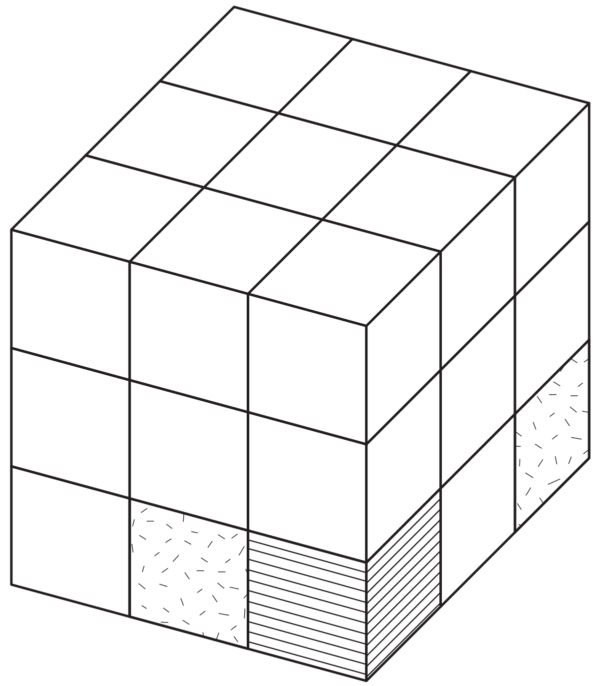
Extension

- Investigate the pattern formed by the number of small cube faces on the outside of the growing cubes: 6, 54 …
- Build a shape from cubes and then photograph or draw the front, side, and top views. Ask another student to draw a grid plan from the pictures. Make the shape from the plan and compare it with the original.

17.1 COLORED CUBES



Use cubes in three different colors to make a square in which each row and column have exactly one cube of each color.



Can you build this cube so that each row and column on every face have exactly one cube of each color?

1. How many cubes of each color did you use? ______

2. (a) What colored cube would be at the center of your larger cube? ______

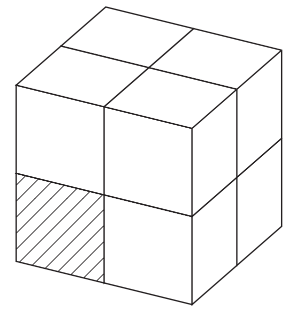
   (b) Can it be any other color? ______

See if you can make an even larger cube from cubes in four different colors and still keep the same pattern.

3. How many of each colored cube do you need? ______

4. Is it possible to continue forming larger cubes in this way? ______

17.2 GROWING CUBES



Place a single cube in front of you. Imagine that you add cubes of a second color to make it a 2-by-2 cube.

1. How many extra cubes would you need? __________

Now imagine that you add cubes of a third color to make a 3-by-3 cube.

2. How many cubes would you need to add to make the 3-by-3 cube? __________
3. How many cubes of a 4th color would you need to add to make a 4-by-4 cube?_______

Get as many of the colored cubes as you think you need to increase the size. See if you can make the 4-by-4 cube you imagined.

4. Explain to a friend how you figured out the number of cubes you needed to increase the size. Did you both do this the same way?

____

____

____

____

____

5. List the number of cubes that you would need to add if you kept growing cubes in this way.

   1 + 7 + ________ + ________ + ________ + ________ + ________

6. Can you describe a pattern for the way the cubes increase in size?

____

____

____

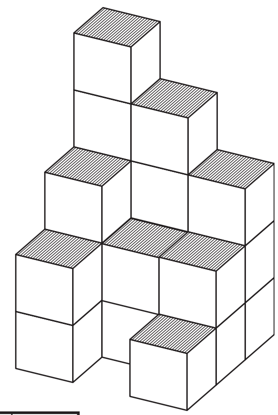
____

____

17.3 VIEWING CUBES

1. Draw what this shape would look like from the:

   (a) front. (b) side.



Use some cubes to make the shape.

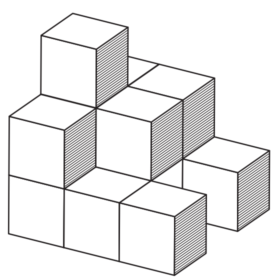
Here is a plan of this shape, using squares to show where the cubes are and numbers to show how many cubes are used.

5	4	3
3	2	2
2		1

2. Does it match the shape you made? ________________

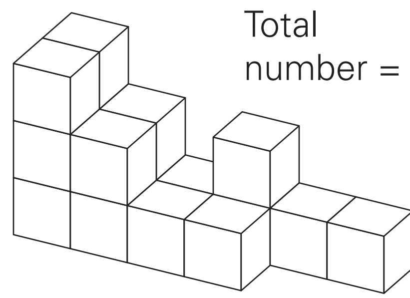
3. Write the number of cubes you would need to make each of these shapes. Get the cubes, make the shapes, and draw a plan for each one.

(a)



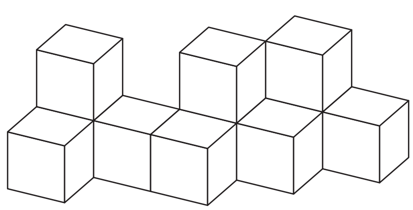
Total number = ________

(b)



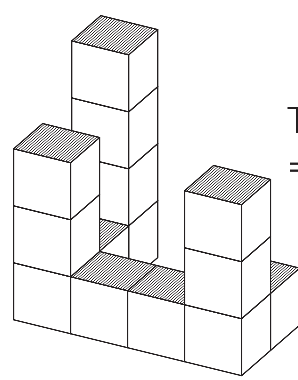
Total number = ________

(c)



Total number = ________

(d)



Total number = ________

4. Make a shape of your own and then make a plan for it. Ask a friend to construct the shape from your plan and compare his or her shape to yours.

Problem-Solving Objective

To use spatial visualization and logical reasoning to solve problems

Materials

Cubes, calculator

NCTM Content Standards

- Geometry 3.1, 3.2, 3.3, 3.4
- Algebra 2.1

Focus

These pages explore arrangements of cubes and longer shapes in order to determine surface areas and volumes and to investigate the patterns they exhibit. Spatial as well as logical thinking and organization are involved as students visualize a given shape in terms of its component parts.

Discussion

Page 79 – Surface Area

Students must visualize how the items are stacked, determine which surfaces are on the outside, and calculate the sum of all the areas. When the builder decides to paint all of the outside faces, there are an additional ten 2 m × 20 cm rectangles to be painted. Care will be needed to ensure all units used are the same (meters expressed as decimal fractions or centimeters). When the 3 × 3 cube is considered, there will be one 1 cm^3 in the middle; when it is removed to create the hole all the way through the cube, the surface area painted will be 6 cm^2 less. The 4 × 4 cube will have a 2 × 2 cube in the middle; the surface area will be 24 cm^2 less when this is removed to make the hole. There are different possibilities for the 5 × 5 cube, depending on whether the hole is 1 cm × 1 cm or 3 cm × 3 cm when cubes are removed to make the same hole from all 6 sides. Students may need to use cubes to see what is happening.

Page 80 –Volume and Surface Area

This set of investigations requires visualizing the arrangement of the cubes in the inside of the prism and then seeing how there is another layer on each side to give the whole prism.

An understanding of prime factors is needed to determine the inside shape. Stacking cubes to form the inside shapes will then help students understand what is required in these problems. In the first problem, 105 has prime factors 3 × 5 × 7, which gives the dimensions of the inside, unpainted shape. Another layer on each side shows that the whole prism is 5 × 7 × 9 and the surface area is

(2 × 35) + (2 × 45) + (2 × 63) or 286 cm^2.

363 has prime factors 3 × 11 × 11 and can be solved in a similar manner.

60 has prime factors 2 × 2 × 3 × 5. These must be considered 3 at a time to give four possible answers:

4 × 3 × 5 2 × 6 × 5

2 × 2 × 15 2 × 3 × 10

Page 81 – Surface Area and Volume

This set of investigations encourages students to see three-dimensional shapes in terms of the component cubes, and figure out the surface area by determining the number of exposed faces and the volume by working out the total number of cubes. The cubes vary in size, so other factors also come into the calculations. Students must work directly from a diagram that shows how the cubes are arranged and from plans that use a grid to indicate the number of cubes in each position. The final investigation requires students to make a shape of their own, construct a plan, and then give the plan to another student to determine the surface area and volume of the shape.

Possible Difficulties

- Unable to visualize the three-dimensional shapes from the information given
- Does not realize the internal prisms are given by considering the prime factors of the number of cubes
- Does not take into account the dimension of the different cubes when figuring out volumes
- Does not consider all of the surfaces when figuring out surface areas

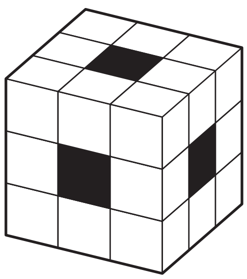
Extension

- Have students make up their own problems involving the surface area and volume of three-dimensional figures and give them to other students to solve.

18.1 SURFACE AREA

1. Some 2 m long wooden beams with 10 cm × 20 cm rectangular ends are going to be stacked to form a set of garden stairs with 5 steps. If all of the exposed sides except those on the bottom and on the back side set into the garden are painted with wood preserver, what area will be painted?

2. The builder decided to paint all of the exposed surfaces of the stairs before he assembled them. What area did he actually paint?



3. Imagine 1 cm^3 cubes stacked to make a 3 × 3 × 3 cube. Remove one cube from the center of each face. If all of the exposed faces are painted red, what is the surface area of the shape that has been painted?

4. Would the surface area to be painted change if there were a hole through the center of the cube? If so, what would it be?

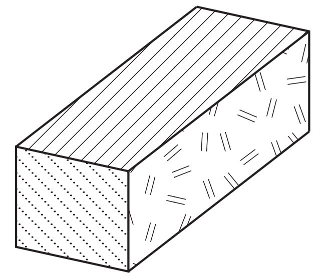
5. (a) Now imagine a 4 × 4 × 4 cube. Remove a square of cubes from the center of each face. What would be the surface area of this shape?

   (b) How would the surface area change if there were a hole in the center of this cube?

6. If you had a 5 × 5 × 5 cube and removed a square of cubes from the center of each face, what surface area could there be? (There are 2 possibilities.)

7. How would the surface area change if there were a hole in the center of the cube? (Consider all the possibilities.)

18.2 VOLUME AND SURFACE AREA



1. A number of 1 cm^3 cubes are put together to make a right rectangular prism with each edge greater than 3 cm. The six faces of the prism are painted green. When the small cubes are taken apart, 105 have no paint on them.

   (a) Where would these cubes occur?

   ____

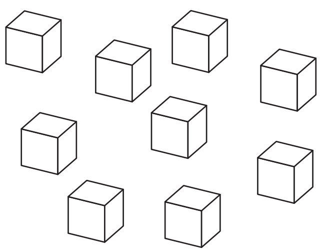
   (b) What shape would they form?

   ____

   Use this information to find:

   (c) the volume of the whole prism. ______________

   (d) the surface area of the whole prism. ______________



2. A large number of 1 cm^3 cubes are put together to make another right rectangular prism with edges greater than 3 cm. After the six faces of the prism are painted green and then the cubes are taken apart, 363 of the small cubes have not been painted.

   What is the volume and surface area of the whole prism?

   ____

   ____

3. Some 2 cm^3 cubes are put together to make a right rectangular prism with edges greater than 6 cm. The six faces of the prism are painted blue. When the small cubes are taken apart, 60 have no paint on them.

   (a) In which ways could 60 cubes form the internal prism?

   ____

   ____

   (b) What would be the volume and surface area of the whole prism for each possibility?

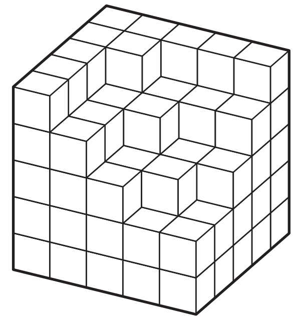
   ____

   ____

   ____

18.3 SURFACE AREA AND VOLUME

1. Some 2 cm³ cubes have been arranged to make this shape.



(a) What is the surface area of the shape?

(b) What is the volume of the shape?

(c) If you add one more block to the upper surface of every block, what will be the new surface area and volume?

6	6	4	2
7	6	4	
6	5	3	
5	4		

2. Here is a plan of a shape made from a number of 3 cm³ cubes. The squares show where the cubes are and the numbers show how many cubes are stacked in each position.

(a) Get the number of cubes you will need and make the shape. Use this model to figure out the surface area and volume of the shape.

(b) Figure out the surface area and volume of the shapes shown by these plans. The size of the cubes is written underneath each plan. (You may need to draw the views from each side and the top to find the surface area.)

(i)

5	4	3	2	1
4	6	6	4	2
3	2	1	1	1

2 cm³ cubes

(ii)

6	3		
5	3		
5	3		
4	3	2	1
4	3	2	1

3 cm³ cubes

(iii)

5	3	2	4
3	5	4	2
2	4	5	3
4	2	3	5

4 cm³ cubes

(c) Make a shape of your own. Then make a plan for it and decide on the dimensions of the cubes. Ask a friend to figure out the surface area and volume. Does your friend need to make the shape?

Problem-Solving Objective

To use spatial visualization and measurement to solve problems

Materials

Paper to make and fold squares and equilateral triangles

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Geometry 3.1, 3.2, 3.3, 3.4
- Algebra 2.1

Focus

These pages explore ideas of perimeter and area, using students' knowledge of squares and triangles to visualize shapes and to determine the lengths of their sides and their areas. Spatial and logical thinking as well as numerical reasoning and organization are required as students investigate the relationships among the shapes to determine the required lengths and areas.

Discussion

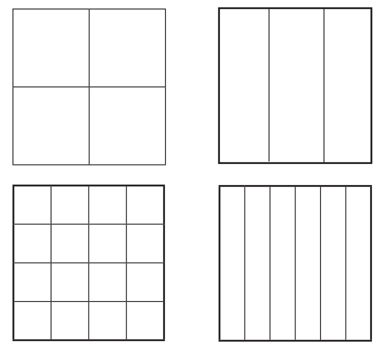
Page 83 – Designing Shapes

Students must be able to visualize the ways in which squares, circles, and triangles relate to one another to give the length of each side. The first shape contains a square with sides measuring one-half that of the large square. In the second shape, the diagonals of the small squares form 8 triangles, so the area of the small square is 1 half the area of the large square. Seven smaller squares form the distance across the large square, so each small square must have sides 4 cm and an area of 16 cm^2. The area and perimeter of the internal shape are then found by summing those of the small squares.

Page 84 – Different Designs

For Problem 1, students need to be able to visualize the shape made of four overlapping triangles, each of which contains four smaller triangles. The area can be determined by figuring out the number of small squares in the shape (13) or by calculating the area of the four large triangles and subtracting the area of the number of small triangles where they overlap. Similar reasoning about the perimeters of the small and large triangles gives the perimeter of the whole shape.

While the next problem can be solved by thinking of the rectangles as being made up of two long sides and two half-length sides to give the four sides for the perimeter of 18 in., this does not readily generalize to the other ways the square is partitioned. Thinking of one rectangle as having six half-sides (or 6 halves) allows the rectangles in Problems 3–5 to be seen as having 8 thirds, 10 fourths, and 12 fifths to readily find the perimeters of the squares. Visualizing how the squares and rectangles are divided into smaller squares allows the results to be easily seen:



Note that thinking of these as fraction symbols (for example, $\frac{2}{3}$ and $\frac{2}{4}$) makes the solution considerably more difficult to determine, since it requires working with fractions rather than whole numbers.

Page 85 – The City Square

Rather than using complex calculation to find the area of triangles on this page, students should be encouraged to search for shapes that can be created from the triangles. For example, in Problem 1, students can form a square from the four triangles with sides 18 m in length. From there, they can calculate the area of the square (18 m × 18 m = 324 m^2), which they can then divide into four to find the area of one triangle (81 m^2). For Problems 2 and 3, students need to closely examine the sections of the smaller squares that are not shaded. They will soon realize that the unshaded areas form triangles whose area can be easily found and subtracted from the total area.

Possible Difficulties

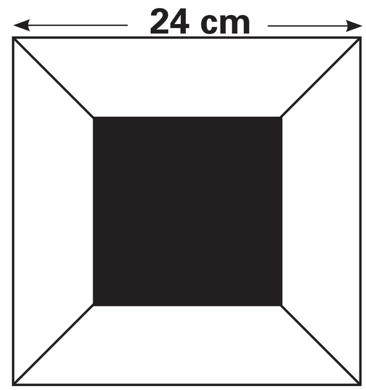
- Unable to visualize smaller shapes within the larger shapes
- Cannot see how areas and perimeters are formed from those of the smaller shapes

Extension

- Have students create similar questions based on finding shapes within shapes.

19.1 DESIGNING SHAPES

A quilter was making a quilt out of squares. The side of each square measured 24 cm in length. To make a pattern for each square, she drew diagonals. She then marked a midpoint between each corner and the middle of the square. Finally, she joined each midpoint, creating a smaller square inside each square (as shown).

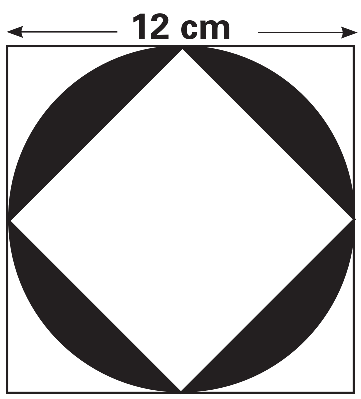


1. (a) The quilter then cut one piece of fabric to cover the smaller square. What was the area of the fabric she used?

   ____

   (b) She then cut other pieces of fabric to the cover the four trapezoid shapes remaining in the larger square. What was the area of each of these pieces?

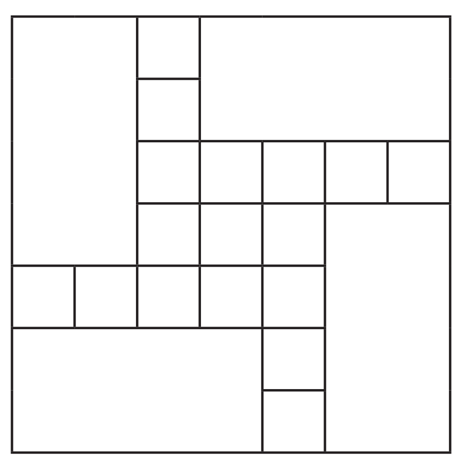
   ____



2. For another quilt, she decided to place a circle inside each square. She then placed a smaller square over the circle. If each side of the larger square measures 12 cm in length, what is the area of the smaller square?

   ____

The quilter then made another design, this time using small squares all the same-size set inside a large square with an area of 784 cm^2.



3. (a) What is the area of the shape made from the small squares?

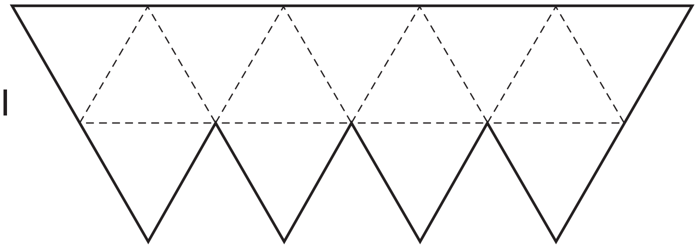
   ____

   (b) What is the perimeter of the shape made from the small squares?

   ____

19.2 DIFFERENT DESIGNS

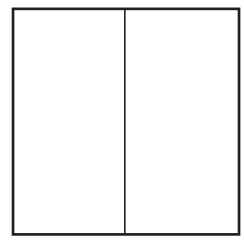
This shape is made from four overlapping equilateral triangles, all the same size. The area of a small triangle (where the large triangles overlap) is 12 in^2.



1. (a) What is the area of a large triangle?

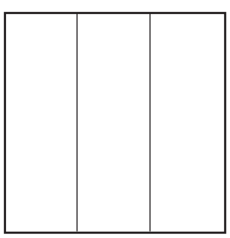
   (b) What is the area of the shape?

   (c) If the perimeter of each of the large triangles used to make the shape is 30 inches, what is the perimeter of the entire shape?

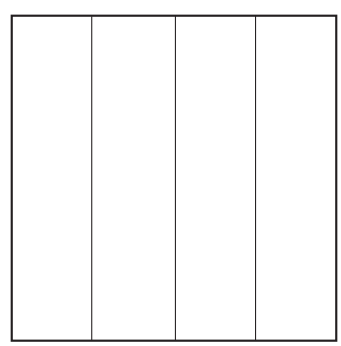


2. (a) A square is cut into two equal-sized rectangles. If the perimeter of each rectangle is 18 inches, what is the perimeter of the original square?

   (b) What is its area?



3. A different square is cut into three equal-sized rectangles, each with a perimeter of 24 in. What is the perimeter and area of the square?

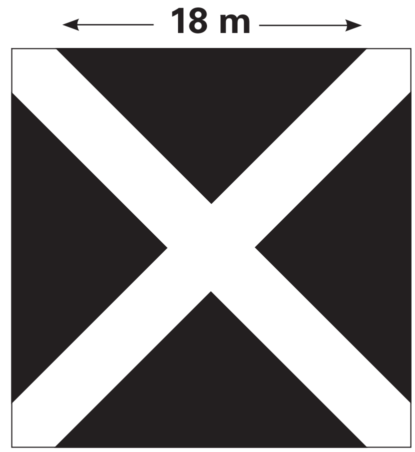


4. Yet another square is cut into four equal-sized rectangles, each with a perimeter of 40 in. What is the perimeter and area of the square?

5. A square is divided into five equal-sized rectangles, each with an area of 180 in^2. What is the perimeter of each rectangle and of the whole square?

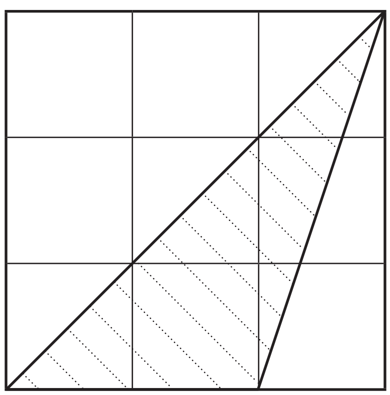
19.3 THE CITY SQUARE

A park in the city was made in the shape of a large square with two diagonal paths 3 m wide and four triangular section of lawn. The length of the long side of each section of lawn is 18 m.



1. What is the area of each lawn?

____

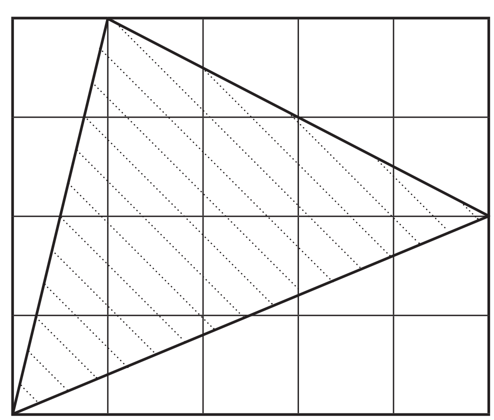


A petition from the city's residents complained that the city square, made from nine smaller concrete squares, each with an area of 36 m^2, was looking tired and dull. To make it look more interesting without having to spend too much money, the council decide to paint a section of the city square green (represented by the shaded areas).

2. What is the area that will be painted green?

____

Residents of the city found the renovated square rather attractive and asked if the plain area in front of the city hall could also be changed. The area was originally made of 20 concrete squares, each 7 m × 7 m. The council decided to place a large area of lawn in the middle of it.



3. What is the area they turned into lawn?

____

Problem-Solving Objective

To use logical reasoning and measurement to solve problems

Materials

Several wooden cylinders, blocks such as Base 10 hundred, calculator

NCTM Content Standards

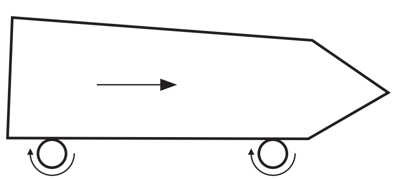
- Number and Operations 1.1, 1.2, 1.3
- Geometry 3.2

Focus

This page explores the use of models in coming to terms with problem situations and analyzing the possibilities that make up the whole solution. Logical reasoning as well as an understanding of circumference are required. Diagrams are also helpful to organize and explore the data.

Discussion

Page 87 – Rolling Along
It will be helpful for students to first model the action of the obelisk moving along with the aid of rollers. If a block is set up on two or more cylinders touching each other, and the point of contact of the block and the front cylinder is marked, it will be possible to see how the block moves forward on the cylinders at the same time as the cylinders move along the ground. For each complete turn of the rollers, the obelisk must move a distance equal to the circumference along the rollers and a further distance equal to the circumference along the ground.



Since the circumference of a roller is 6.25 m, the obelisk will move forward 12.5 meters with each rotation of the rollers. Moving 3.2 kilometers or 3,200 meters requires 3,200 ÷ 12.5, or 256 rotations. Each day the obelisk moves 12 rotations, or 150 m. After 21 days, it would have moved 3,150 m, and after 22 days, it would have moved 3,300 m. The obelisk would have moved 3,200 m on the 22nd day after Friday—a Saturday.

The experience with the rollers will help in imagining how the gears move: gear B would move in a clockwise direction and gear C in an counterclockwise direction. gear C has 3 × 48 teeth, so gear B would make 6 turns and gear A would make 4 turns to engage this many teeth.

When gear A rotates 12 times to lower 216 m, 12 × 36 teeth would need to engage. Gear C would need to rotate 9 times to lower the load (9 × 48 teeth would need to engage).

Possible Difficulties

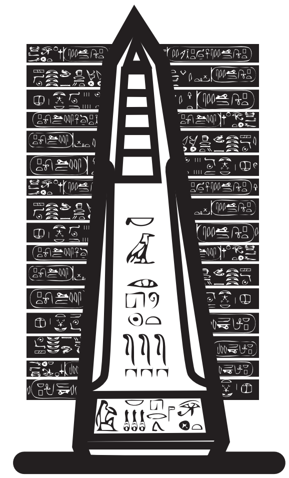
- Unable to see how the obelisk will move a distance along the rollers and also a distance along the ground
- Not seeing that division is required to find the number of times the rollers rotate
- Not realizing that the obelisk would not require a whole number of days to reach the Nile and would be there during the 22nd day
- Unable to see that 22 days is three weeks and one day, so it must be one day after a Friday or a Saturday that it reached the Nile

Extension

- Ask how far the obelisk will have traveled after 5, 10, or 15 days.
- Pose similar problems to the one on the page where large granite blocks are moved on rollers with different circumferences and different distances to reach the pyramids and other large Egyptian constructions.
- If students are familiar with calculating circumferences using the ratio π, pose similar problems where the radius of the roller is given.
- Change the number of teeth on each of the gear wheels (they need to have factors in common to work).
- Have students investigate the use of rollers, block and tackles, and gear levers in other situations—for example, on cranes, boats, and winches.

20.1 ROLLING ALONG

1. (a) The ancient Egyptians made their obelisks by cutting and shaping large granite pieces from a quarry near Aswan. Log rollers were used to move them to the Nile River so they could be floated to where they were needed at Luxor. If the circumference of each roller was 6.25 m, how far did the obelisk stone move for each complete turn of the rollers?



____

(b) How many times would the rollers have to rotate to move the obelisk the 3.2 km to the Nile River?

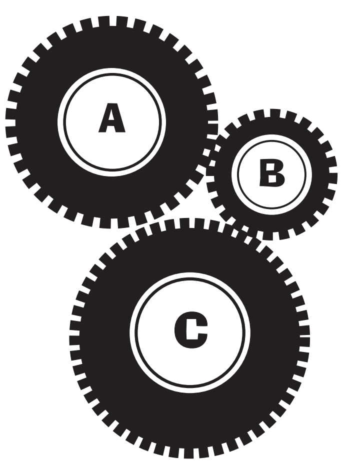
____

(c) The workers were able to turn each roller through 12 rotations per day. If they started moving the obelisk on a Friday, on which day did it reach the Nile River?

____

2. In the time of the ancient Egyptians, a pulley was used to lift relatively small objects, since it is easier to pull down on a rope than to lift up with a rope. These days, systems of pulleys or geared wheels are used on cranes that lift and lower large objects so they can be swung into position rather than rolled into place.

In this system, gear A has 36 teeth, gear B has 24 teeth, and gear C has 48 teeth.



(a) When gear A turns in a counterclockwise direction, in which direction do gears B and C turn?

____

(b) When gear C makes 3 complete turns, how many turns do gears B and A make?

____

(c) If each rotation of gear A lowers a heavy load 18 m, how times would gear C need to rotate to lower the load 216 m?

____

Problem-Solving Objective

To solve problems involving time and make decisions based on particular criteria

Materials

Clock

NCTM Content Standards

- Number and Operations 1.2, 1.3
- Geometry 3.2

Focus

This page explores reading for information, obtaining information from a number of sources (the information about the plane, timetable, and shuttle bus), and using it to find solutions. The problems involve thinking about and working with time. Decisions about time being too early or too late are needed rather than an exact time.

Discussion

Page 89 – Red Rock Adventures
Students read the information on the page and use it to solve problems. Students must read for information from a number of sources and match this information to set criteria.

A number of flights fit the criteria, with other flights arriving in Utah either too early or too late. Once a flight has been deemed too late, then others that are later still can be automatically excluded; for example, if the Desert Sagebrush flight at 3:10 p.m. would not arrive in Utah before dinner, then the flights at 3:25, 4:50, and 5:45 p.m. can also be ruled out and no longer need to be considered.

Similar thinking can be used for the flights that are too early; for example, catching the Red Center flight at 9:45 a.m. would get you to the hotel at around 1:05 p.m., which is too early. As such, all flights prior to this time would also be too early and can be quickly excluded.

In Problem 5, the information regarding the taxi and the wait time at the airport is needed to determine what would be the latest possible time to leave home and still arrive at the airport on time. Some students may try to include this in their travel calculations. The bus leaves the airport every 15 minutes starting at 7:00 a.m., which means a bus leaves on the hour and 15, 30, and 45 minutes past each hour. However, if the plane arrives at 2:15 p.m., they would not be able to catch the 2:15 p.m. bus, since they need to first exit the plane and collect their luggage. Students must consider this when solving the problem.

Possible Difficulties

- Unfamiliarity with a timetable
- Including taxi and wait time information in their travel calculations
- Thinking that an exact flight is needed rather than flights that are neither too early nor too late

Extension

- Use the information and timetable with other criteria—for example, if you need to be in Utah for a sunset tour or mid-morning hike.

21.1 RED ROCK ADVENTURES

Imagine you have decided to travel from Calgary to Canyonlands National Park in Utah for a vacation.

Look at the following information and plan your vacation. As most hotels have a 2 p.m. check-in time, you want to arrive after 2 p.m. but before dinner at 6:30 p.m.

TAXI INFORMATION – from home to Calgary Airport

- wait time: 25 min
- travel time: 75 min

PLANE INFORMATION

- wait time at airport: 1 hr
- flight time to Salt Lake City, Utah: 2 hr 50 min

Red Center Airlines

	Flight numbers					
	RC23	RC32	RC47	RC51	RC68	RC74
Departure times	7:20	9:45	12:35	1:50	3:25	5:45

Desert Sagebrush Airlines

	Flight numbers					
	DS23	DS32	DS47	DS51	DS68	DS74
Departure times	7:10	9:25	11:40	2:15	3:10	4:50

BUS INFORMATION – from Salt Lake City airport to hotel

- hotel bus leaves airport: every 15 minutes, starting at 7:00 a.m.
- travel time to hotel: 20 min.

1. I can catch the following flights: ____________________
2. These flights are too early: ____________________
3. These flights are too late: ____________________
4. What would be the latest time I can leave home for each possible flight?

____

5. What flight would you choose to get you to your destination on time and why?

____

Problem-Solving Objective

To use logical reasoning and measurement to solve problems involving a balance, money, or a calendar.

Materials

Counters, calculator, calendar

NCTM Content Standards

- Number and Operations 1.2, 1.3
- Geometry 3.2
- Data Analysis and Probability 4.1

Focus

These pages explore different ways of solving problem situations and analyze the different aspects that contribute to the solution. Logical reasoning as well as an understanding of measurement when using money or a calendar are required. In each situation, diagrams can be used to organize, sort, and explore the data.

Discussion

Page 91 –Calendar Calculations

In Problem 1, putting the three people's work rosters into a table shows when they work and when their days coincide (Saturday of week 5):

Week	Mon	Tues	Wed	Thur	Fri	Sat	Sun
1	S			S			S
		J				J	
				P			
2			S			S	
			J				J
		P					P
3		S			S		
				J			
					P		
4	S			S			S
	J				J		
			P				
5			S			S	
		J				J	
	P					P	

Problem 2 requires students to divide the 300 days by the number of days there are each week (300 ÷ 7) to determine the number of weeks' difference there is between the two birthdays. However, this provides a result of approximately 42.86, which is not practical. It is then best to discuss the result as 42 weeks and six days (42, remainder 6). Students can then use this to determine that the day of the week is a Thursday.

In Problem 3, students must determine that for there to be five of any day of the week in a February, there have to be more than 28 days. (28 can be divided by 7 without there being any remainder.) Therefore, for there to be five Fridays during one February, it must be a leap year (29 days). 2008 had five Fridays, while the next leap year, 2012, has five Wednesdays.

Page 92 – Balancing Business

Both Problems 1 and 2 use the concept of a balance to rule out possibilities. Students are encouraged to look for an efficient means of solving the problems rather than completing the tedious task of comparing the boxes of screws or bags of nails one at a time. In Problem 1 there must be 23 full boxes on each side. Dividing 850 by 23 gives 36, with a remainder of 22. Each full box has 36 screws, and the partly filled one has 22. In the second problem, Harry should divide the bags of nails into groups of three to quickly determine the bag that is heavier or lighter.

Page 93 – Puzzle Scrolls 2

These problems all require strategic thinking to find possible solutions. In many cases, students will find that tables, lists, and diagrams are needed to manage the data while exploring the different possibilities.

Possible Difficulties

- Thinking that each bag of nails has to be weighed one at a time
- Not realizing that 28 days is a multiple of 7 days
- Difficulty keeping track of numbers in the strategic thinking questions

Extension

- Change the roster of days the staff works.
- Have students make up their own problems based on the calendar and days in a week.
- Find some of the famous problems that use a balance to deal with different amounts of counterfeit coins or "fool's gold."

22.1 CALENDAR CALCULATIONS

1. Three friends work part-time in a café that is open 7 days a week. Simon is able to work once every 3 days; his girlfriend, Jackie, works once every 4 days; and Jackie's sister, Penny, works 1 day and then has 4 days off. This week, Simon worked on Monday, Jackie worked on Tuesday, and Penny worked on Thursday. They like it best when they work together at the cafe on a Saturday. How many weeks will it be before that occurs?

   ____

2. Today is my friend's birthday. Her birthday is almost 10 months after mine. Actually, it is exactly 300 days later. If my last birthday was on a Friday, what day of the week is today?

   ____

3. During certain years in the 21st century, February has exactly five Fridays. What type of year does it have to be for this to happen and why?

   ____

   ____

4. What day of the week would next occur five times in February?

   ____

5. One year, July had only four Fridays. On what days of the week could July 31 have occurred that year?

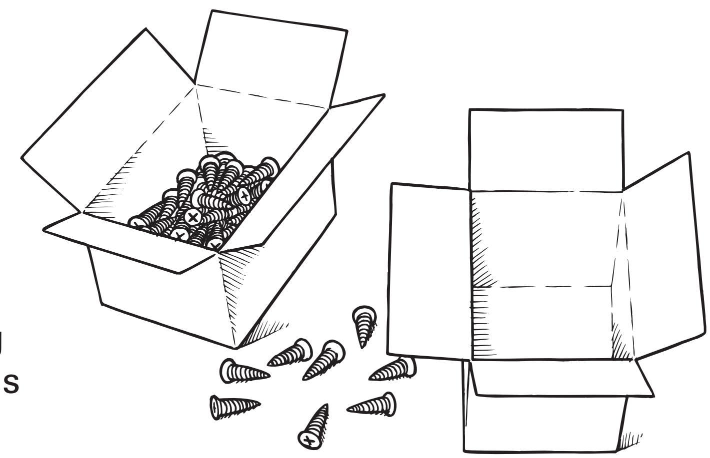
   ____

22.2 BALANCING BUSINESS

Harry and Hugh work in their father's hardware store during school vacations. Harry's first job was to sort all of the shop's screws into equal amounts, which were then to be boxed. He found 23 full boxes, one partly filled box, 850 loose screws, and 24 empty boxes. Unfortunately, the boxes were not labeled to state how many screws each should contain. (Assume the screws were all identical and the boxes the same size.)

1. Harry decided to use a balance to find the number of screws each box should have and then fill all the boxes. The 23 full boxes and one partly filled box balanced the 850 loose screws and 24 empty boxes. How many screws are in each full box?

Hugh's job was to pack bags of nails for sale. There was one filled bag on the shelf, and he balanced it with a bag and a pile of loose nails.



When he took a break, Harry tried to help by filling a bag for him. Unfortunately, Harry was not paying attention, and the bag he packed was heavier than it should be.

No one noticed until a total of 78 bags were filled. When Harry saw Hugh filling the last bag, he realized his mistake and tried to fix it by using the balance to find the heavier bag.

2. How could he do this using the balance as few times as possible?

22.3 PUZZLE SCROLLS 2

1. I am less than 2 m tall but more than 1 m tall. My height in centimeters is a multiple of 6, 7, and 8. What is my height?

2. Two days before yesterday was Thursday. What day of the week will it be in two weeks' time?

3. A light flashes every 2 minutes, and another flashes every 3 minutes. They flash together at 8:00 a.m. What time will they next flash together?

4. You enter an elevator and go up 9 floors, down 3 floors, and up 5 floors. You get out at floor 13. At what floor did you enter the elevator?

5. A car travels 2 miles in 2 minutes and 16 seconds. How far will it travel in one hour?

6. The sum of Anthony's and Alan's heights is 146 in. Alan is 9 in. taller than Anthony. How tall is Anthony?

Problem-Solving Objective

To analyze and determine probability

Materials

Calculator

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Data Analysis and Probability 4.2, 4.3, 4.4

Focus

These pages explore word problems that focus mostly on probability. Students must determine what the problem is asking and, in many cases, carry out more than one step to find a solution. Analysis of the problems reveals that some contain information that is not needed. A calculator can be used to assist if necessary, since these problems are about determining what the problem is asking rather than computation or basic facts.

Discussion

Page 95 – World Cities Weather
This page asks students to read information in a table and use that information to determine means and probabilities. The table lists a summary of weather information for various cities in the world. Estimation is important at this stage so that students don't simply calculate every combination to find a solution. The investigation about the monthly average rainfall requires students to take the total amount of rain and divide it by the number of days it rained. This will result in a daily average. Discussion could focus on what a daily average tells us, as in many cases the daily average would not be how much rain actually fell on those days.

Page 96 – Showtime
The probability of an event occurring is a number describing the chance that the event will happen. An event that is certain to happen has a probability of 1. An event that cannot possibly happen has a probability of zero. If there is a chance that an event will happen, then its probability is between zero and 1. Probability is determined by taking the number of chances of the event occurring and dividing it by the total number of chances. It can be expressed as a percent or as a ratio.

Some investigations on this page use terminology such as "winning streak" and "bad luck," which may influence a person's thinking but has no effect on probability.

The first two problems look at the probability of Marcy's three throws totaling various amounts. Students must consider the possible results for all three throws and add the points. The problem about Krista losing four times in a row explores sequence probability and involves multiplying five-sevenths four times to work out the probability of four losses in a row.

Page 97 – Probably True
These pages further explore the concept of probability begun on page 95. If an event has a probability of 0, then it cannot happen, while a probability of 1 means it is certain to happen.

As discussed above, probability is determined by taking the number of chances of an event occurring and dividing it by the total number of chances. For example, selecting a red marble from a bag involves finding out how many red marbles are in the bag (4) and dividing it by the total number of marbles (12) to get a ratio of 4:12, which can be also written as 1:3.

Possible Difficulties

- Confusion over the concepts of chance and probability
- Taking into consideration previous events that are not related to the current event
- Difficulty with the concept of ratio

Extension

- Look at the temperature and number of days of rain for the previous year and explore the probability of rain, sunshine, and temperature for your local area.
- Explore how vocabulary is used when discussing chance and probability.

23.1 WORLD CITIES WEATHER

City	Average High Temp (°F)	Average Low Temp (°F)	Rainfall (in.)	Days of Rain
Amsterdam	72	34	29.9	175
Athens	92.3	41.4	15	87
Mumbai	92.1	61.5	86.6	80
Copenhagen	68	28.4	25.8	168
Istanbul	83.3	37	27.2	124
London	72.1	36.3	23	145
Madrid	88.2	36.7	16.5	95
Tokyo	87.4	35.8	55.4	101

1. Which city has the largest temperature range? ______________________
2. What is the probability of rain on any day in Amsterdam?

____

3. Which city has a mean temperature of 54.2° F? ______________________
4. What is the probability of it not raining in Madrid?

____

5. Which city has an average of about 7.2 inches of rain per month?

____

6. Which five cities have a mean temperature of more than 60° F?

____

7. Which city has the lowest mean temperature?

____

8. What is the probability of the day being sunny in Istanbul?

____

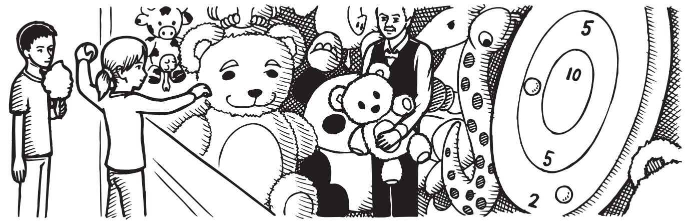
9. What is the probability of the day being rainy in Athens?

____

10. What is the average amount of rain that falls on a rainy day in London?

____

23.2 SHOWTIME



1. Marcy is playing "Three Best Throws" in Side Show Alley. She has to throw 3 balls at a circular target that automatically totals the points she gets. If a ball hits the center, she gets 10 points. The middle region scores 5 points and the outside 2 points. Assume that all her throws hit the target, and she is equally likely to hit any point on the target. What is the probability that she will score 20 points or more?

2. What is the probability she will throw a winning score of 25?

3. Marcy has a winning streak, and on the next 2 throws she wins a prize. What is the chance her next throw will also win a prize?

4. Krista is playing "Pick the Prize Card," in which she has to pick a card from a set of 7 cards to win a prize. Each set of 7 cards has 2 winning cards. She has had 5 tries and has won 2 times. In her next turn what is the likelihood she will pick a prize card?

5. Unfortunately, Krista loses on her next 3 turns. What is the probability of her losing 4 times in a row?

6. Rodney works at the hot dog stall from 10:30 a.m. to 12:30 p.m. each day. The stall sells hot dogs for $3.00 each, 2 for $5.00, or 4 for $8.00. Assume that a customer is equally likely to choose 1, 2, or 4 hot dogs. Rodney serves 7 customers in a row who all want hot dogs at 2 for $5.00. What is the probability his next customer will want 2 hot dogs for $5.00?

7. What is the probability he will serve 9 customers in a row who want 2 hot dogs for $5.00?

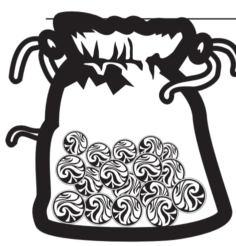
8. Would he be more likely to sell 9 single hot dogs in a row?

23.3 PROBABLY TRUE

1. Georgia is playing cards using a regular pack of 52 cards. How many cards would she need to draw from the pack to be absolutely certain she will draw a card with diamonds on it?

2. One bag contains 3 blue dice and 2 yellow dice, and another bag contains 2 blue dice and 1 yellow die. To win a prize, Tony needs to select a blue die from either bag. He can only have one try. Which bag should he choose and why?

3. Andy and Pete are playing with two 10-sided dice with the digits 0–9 on them. When they roll the dice, they add the numbers together. What is the probability they will get a multiple of 6?



4. A bag has 5 blue marbles, 3 red marbles, and 4 yellow marbles. How many red marbles must be added to the bag so that the probability of drawing a red marble is $\frac{3}{4}$?

5. A jar contains 2 yellow marbles, 3 red marbles, and 4 blue marbles. Carly draws one marble from the jar, and then Toby draws a marble from those remaining. What is the probability that Carly will draw a blue marble and Toby a red marble?

6. June and Shayla have a bag with 5 green counters numbered 1–5. At the same time, they each pick a counter. June calculates the sum of the numbers while Shayla calculates the product. What is the likelihood that the sum is greater than the product?

7. Kurt is playing with a tetrahedron die with the digits 1–4 on the four faces. If he rolls the die twice, what is the probability that the same number will be on the bottom face each time?

Problem-Solving Objective

To analyze information and use proportional and logical reasoning to solve problems.

Materials

Calculator

NCTM Content Standards

- Number and Operations 1.1, 1.2, 1.3
- Data Analysis and Probability 4.1, 4.2

Focus

These pages explores problems that require an ability to carefully analyze the relationships among several different items of data and use an understanding of proportional reasoning and numbers to organize the information and to keep track of the possible answers. Pages 99 and 100 are to be used in conjunction. The problems on page 101 introduce the idea of arranging the various interrelated numbers into a diagram that allows all of the overlapping conditions to be visualized and dealt with in a systematic way.

Discussion

Page 99 – Tank Water

While a week has seven days, a working week varies according to the type of work and the expectations of the workers. In the first problem, each of the four workers worked for three weeks and three days (18 days). This means the job requires 4 × 18 or 72 days' work. If only three workers are available, each works for 24 days, and the job would take four weeks and four days to complete (72 days' work in total; 24 × 3 = 72). Some students may also be able to reason that when a team of four workers is replaced by a team of three workers, it would take four-thirds of the time (24 days). Counters or Base 10 materials could also be used in an array to show how 4 × 18 can be rearranged as 3 × 24.

Page 100 – Square-Deal Nursery

There are several ways these problems can be solved. For Problem 1, only 3-by-3 or 4-by-4 trays have been sold. Since 927 plants were sold, they could not all be 3-by-3 trays because that would mean 103 trays had been sold rather than 68. This is 35 trays too many. Since the larger tray holds seven more plants than the 3-by-3 tray, the excess number of plants can be divided by this number to determine 45 4-by-4 trays (315 ÷ 7 = 45). If this amount is taken from the total amount of trays (68 – 45), the result is 23; therefore, 45 4-by-4 trays and 23 3-by-3 trays were sold (a total of 927 plants).

Another method is to put multiples of 9 or 16 into a table and systematically check and adjust until a solution is reached. Counters could be used to explore ways to solve the problems using smaller numbers. In Problem 3, finding the numbers of trays recycled requires the total cost to be divided by the cost of the trays.

Page 101 – Salad Days

These problems involve a large amount of overlapping data that must be sorted. A Venn diagram is used to help solve the problem. The statements must be read carefully to see where the numbers are placed so the answers can be identified: 65 can be placed in the middle area, where all possibilities are included. That means the section shared only by peppers and feta must have 15, since 80 salads had feta and peppers. Also, the area shared only by olive and feta must have 45, since 110 had olives and feta. Since 230 salads had olives or feta, 20 must have had only peppers, so the remaining area in the peppers circle must be 20.

Since 75 salads did not have olives, the bottom area of feta must be 40. The last condition that 115 salads did not have peppers means that 135 salads did have peppers and is the key to completing all the entries. From this, we can find that 35 goes in the area shared only by olives and peppers.

Finally, the last area of olives must have 30 to make the total 250. Students may find this difficult to achieve at first. Covering the areas that are not being considered with their thumb or finger may help them see what to do.

Possible Difficulties

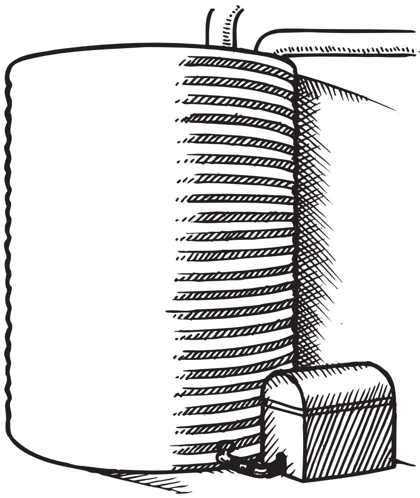
- Not seeing how proportion applies to the problems on page 99
- Working with the numbers in the problem rather than the differences on page 100
- Unable to sort out the layers of overlapping information on page 101

Extension

- Have students investigate the history of logic problems and the work of John Venn and Lewis Carroll (mathematician and author of *Alice in Wonderland)*.

24.1 TANK WATER

Under a new city law, every house built in a new Las Vegas housing development must have a rainwater tank installed. A team of 4 workers took 3 weeks and 3 days to install one large tank for each of the 36 houses in one street. For the next street, which also has 36 houses, only 3 workers are available.



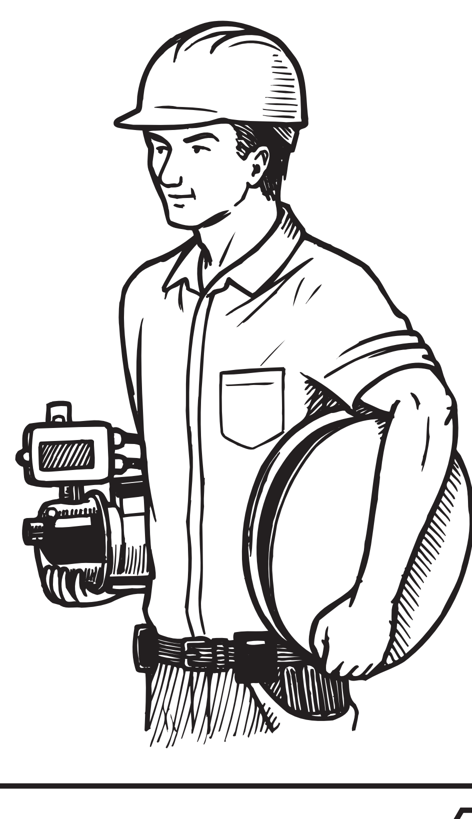
1. How long will it take the team of 3 workers to do the job if they all work at the same rate? (The team does not work on weekends.)

    ____

Plumbers are needed to connect the water tank to the showers and toilets in each house. At first, it seems that they will only be able to get 2 teams of 4 plumbers to connect the tanks, taking an estimated 4 weeks and 3 days to finish. However, at the last minute, they find another team of 4 plumbers.

2. How long will it take the 3 teams of 4 plumbers to complete the connections? (The plumbers work 6 days per week.)

    ____



The last job in the project requires electricians to install a pump for each tank. Working 5 days per week, 3 teams of 2 electricians are expected to take 5 weeks to install the pumps.

3. If there were only 3 weeks to do the job, but the electricians agreed to work 6 days a week, how many teams of 2 electricians would be needed?

    ____

24.2 SQUARE-DEAL NURSERY

1. With all of the water tanks installed, the new homeowners were able to establish gardens. The local nursery supplied plants in square trays of various sizes. The trays had grids of 2×2, 3×3, or 4×4, depending on the size. During the first weekend, they sold 927 plants in 68 trays, some of which were 3×3 trays and the rest 4×4.

   How many 9-plant and 16-plant trays did they sell?

   ___

2. Some months later, larger plants were requested for the backyards and only 2×2 and 3×3 trays were sold. At the end of the weekend, 1,470 plants had been sold and 300 trays had been used.

   How many 9-plant trays were sold?

   ___

3. To encourage the recycling of plant trays, $5 was refunded on a 3×3 tray, $3 on a 2×2 tray, and $7 on a 4×4 tray. Last week, $896 was refunded to people who returned trays, with twice as many 2×2 and three times as many 3×3 trays returned as 4×4 trays.

   How many of each type of tray were recycled?

   ___

4. A drip irrigation system with three electronic timers costs $9,600. If the irrigation equipment costs $7,500 more than the timers do, what is the cost of one electronic timer?

   ___

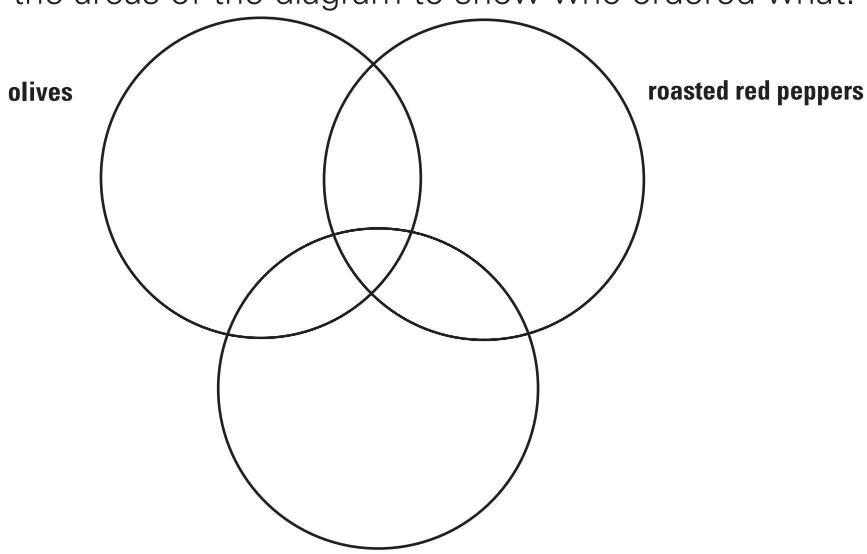
24.3 SALAD DAYS

At Salad Shack, all of the salads come with lettuce and tomatoes, but customers can choose to have one or more of feta cheese, olives, and/or roasted red peppers. Each salad has at least one of these items. On Sunday night, 250 salads were sold between 5 p.m. and 6 p.m.

- 75 of the salads did not have olives.
- 230 of the salads had olives or feta cheese.
- 65 of the salads had roasted peppers, olives, and feta cheese.
- 80 salads had roasted red peppers and feta cheese.
- 110 salads had olives and feta cheese.
- 115 salads did not have roasted red peppers.

Use this diagram to show how many people ordered each of the different types of salads.

1. Fill in each of the areas of the diagram to show who ordered what.



This is known as a Venn diagram and was introduced to solve complex logical problems of this form by the mathematician John Venn in the 19th century.

2. How many salads had olives and roasted red peppers? ____________

3. How many salads had only olives? ____________

4. How many salads had only feta cheese? ____________

SOLUTIONS

Note: *Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

HOW MANY? .... page 25

1. 115,500
2. 368,300
3. 81,437
4. 46,270

HOW FAR? .... page 26

1. 55,500 m
2. about 191.1 mi
3. 68 km
4. 228 mi
5. about 571.1 mi

HOW MUCH? .... page 27

1. He buys 7 × 10 lb blocks of cheddar @ $96.50 each.
   He sells 4 × 68 packs @ $4.50 each.
   Profit is $548.50.
2. Hourly rate is $23.50; leaflet rate of $29.65 pays more.
3. 168 melons
4. With screws is $198.80; without screws is $196.40—$2.40 cheaper.
5. $184

THE SEEDLING NURSERY .... page 29

1. 2,586
2. 1,056
3. (a) $3,886 (b) $2,412 profit
4. 460
5.

3 × 9 = 27 53 × 4 = 212	7 × 9 = 63 44 × 4 = 176	11 × 9 = 99 35 × 4 = 140	15 × 9 = 135 26 × 4 = 104	19 × 9 = 171 17 × 4 = 68
23 × 9 = 207 8 × 4 = 322				

6.

4 × 6 = 24 44 × 8 = 352	8 × 6 = 48 41 × 8 = 328	12 × 6 = 72 38 × 8 = 304	16 × 6 = 96 35 × 8 = 280	20 × 6 = 120 32 × 8 = 256
24 × 6 = 144 29 × 8 = 232	28 × 6 = 168 26 × 8 = 208	32 × 6 = 192 23 × 8 = 184	36 × 6 = 216 20 × 8 = 160	40 × 6 = 240 17 × 8 = 136
44 × 6 = 264 14 × 8 = 112	48 × 6 = 288 11 × 8 = 88	52 × 6 = 312 8 × 8 = 64	56 × 6 = 336 5 × 8 = 40	60 × 6 = 360 2 × 8 = 16

THE TROPICAL ORCHARD .... page 30

1. 235 trees and 6 days
2. 2 hectares
3. (a) 5,664 mangoes
   (b) Between 336 and 342 mangoes are rejected – approximately 340.
   (c) 2,574 kg of bananas
4. 2,900 plants

ANIMAL SAFARI PARK .... page 31

1. 390 bags
2. The fish lasts 17 days, with 2 boxes left for the next day.
3. 108.5 bales cost $1,302.
4. 720 kg
5. 1,320 cars
6. 207 more people
7. 276 people seated

AT THE MALL .... page 33

1. (a) $1,123.25
   (b) $1,995.00
2. (a) $112
   (b) 3 DVDs for $40.50 and 2 CDs for $42.50
3. (a) 6 T-shirts for $100 and 2 pairs of cargo shorts for $45
   (b) $137.75
   (c) $140

AT THE DELI .... page 34

1.

boxes of 16 cans	number of cans	number of jars	boxes of 12 jars	total boxes
1	16	100	not possible	
2	32	84	7	9 too many
3	64	52	not possible	
4	64	52	not possible	
5	80	36	3	8

   3 boxes of jars were unpacked.
2. (a) $15 profit per tub
   (b) 4 tubs
   (c) when February has exactly 4 weeks, about $144 profit
3. (a) 3 tubs cost $28.50, loose feta costs $29.40, 3 tubs are cheaper.
   (b) Profit on 1 kg of loose feta is $9.20.
   There is $128.80 more profit per week in the summer.

THE SUGAR MILL .... page 35

1. 816 bins (815 full, 1 part full)
2. maximum number of trains 7 (with 135 bins)
   minimum number of trains 6 (with 145 bins)
3. Harvesting time is 7 months – approximately 2,123 metric tons per month and 1,061.5 metric tons per farm.

SOLUTIONS

Note: *Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

4. (a) 4,850 bins were used; between 34 trains (if all have 145 bins) and 36 trains (if all have 135 bins)
   (b) about 3,675.5 metric tons
5. 353 truckloads were needed: 88 trips with 4 trucks, and 1 trip with 1 truck.

BOOKWORMS .... page 37

1. 124 pages
2. Chapter 8
3. 27 days
4. 1241–1364
5. 9 and 10
6. 558
7. 882
8. 792
9. 558
10. 9

PROFIT AND LOSS .... page 38

1.

Items sold per week	50	75	100	250	500	1,000	1,500	3,000
Income	$2,500	$3,750	$5,000	$12,500	$25,000	$50,000	$75,000	$150,000
Total costs	$4,750	$5,125	$5,500	$ 7,750	$11,500	$19,000	$26,500	$ 49,000

2. Loss
3. Profit
4. Between 100 and 250

	110	113	114	115
Income	$5,500	$5,650	$5,700	$5,750
Fixed costs	$4,000	$4,000	$4,000	$4,000
Variable costs	$1,650	$1,695	$1,710	$1,725
Total costs	$5,650	$5,695	$5,710	$5,725

115 items

CALCULATOR PATTERNS .... page 39

1. (a) 4,899
   (b) 4,900
   (c) difference of 1
   (d) Numbers will vary, but difference will always be 1.
   (e) Numbers will vary, but difference will always be 1.
   (f) For consecutive numbers, the product of the number before and the number after is 1 less than the number squared, number^2.
   The number before is (number – 1) and the number after is (number + 1).
   (number – 1) × (number +1)
   = number × number + number × 1 – 1 × number – 1 × 1
   = number^2 + number – number – 1
   = number^2 – 1
   No matter what number is chosen, there will always be a difference of 1.
2. (a) 279
   (b) 93
   (c) 3
   (d) Try 83
   The difference between 8^3 and 3^3 is 485.
   $8^2 + (8 \times 3) + 3^2 = 97$
   $485 \div 97 = 5$
   (e) Answer is always the difference between the tens and ones digits.

MAGIC SQUARES .... page 41

1. 27
2. (a)

16	**36**	**8**
12	20	28
32	**4**	24

60

(b)

45	3	**33**
15	27	**39**
21	51	**9**

81

(c)

28	**63**	**14**
21	35	**49**
56	7	42

105

(d)

75	**5**	55
25	45	**65**
35	**85**	15

135

3. (a)

22	16	10	**42**
12	40	24	**14**
38	6	**20**	26
18	28	**36**	8

90

(b)

30	**21**	12	**51**
15	48	**33**	18
45	**6**	27	36
24	39	**42**	9

114

(c)

28	22	**16**	46
18	**44**	30	**20**
42	12	**26**	32
24	**34**	**40**	14

112

4.

9	**16**	3	15	22
5	12	**24**	6	**18**
21	**8**	20	2	14
17	4	**11**	**23**	10
13	**25**	7	19	**1**

65

SUDOKU .... page 42

1. (a)

6	1	2	5	3	4
5	3	4	2	6	1
3	2	5	4	1	6
1	4	6	3	2	5
4	6	3	1	5	2
2	5	1	6	4	3

(b)

5	3	6	1	4	2
1	4	2	5	6	3
3	6	5	2	1	4
4	2	1	6	3	5
2	1	3	4	5	5
6	5	4	3	2	1

SOLUTIONS

Note: *Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

(c)

5	7	2	1	9	4	8	3	6
8	3	1	7	6	2	5	9	4
4	9	6	5	3	8	7	2	1
9	1	8	6	2	5	3	4	7
7	6	5	8	4	3	9	1	2
3	2	4	9	1	7	6	8	5
1	4	9	3	5	6	2	7	8
6	8	3	2	7	1	4	5	9
2	5	7	4	8	9	1	6	3

(d)

8	9	7	5	4	1	2	3	6
6	1	4	3	7	2	8	5	9
3	5	2	9	8	6	4	1	7
1	6	8	4	2	3	7	9	5
5	4	3	7	6	9	1	2	8
2	7	9	1	5	8	6	4	3
4	2	5	8	3	7	9	6	1
9	8	6	2	1	5	3	7	4
7	3	1	6	9	4	5	8	2

ALPHAMETIC PUZZLES .... page 43

1. (a)

	9	5	6	7
	1	0	8	5
1	0	6	5	2

(b)

		8	9
1	9	4	6
2	0	3	5

There are many possible solutions to this alphametic puzzle. Some others are:

49	89	49
1,986	1,947	1,987
2,035	2,036	2,036

(c)

		8	0	7
3	9	2	2	8
4	0	0	3	5

(d)

			5	7
		8	7	0
	9	4	4	1
1	0	3	6	8

NUMBER PATTERNS 1 .... page 45

1. 16
2. (a) 36
   (b) 100
3. The sum of a certain number of the first odd numbers is that number squared.
4. 25
5. 144
6. It increases by 2 each time.
7. They are square numbers.
8. at the beginning of the 11th row
9. 183 is greater than 13^2 or 169 and less than 14^2 or 196, so it would be in the 14th row.
10. 168 is the 2nd to last number in row 13. The number beneath it would be the 3rd to last number in row 14, or 194.
11. The number at the end of the first row of the triangle is the first square number. The number at the end of the second row is the second square number. The number at the end of the third row is the third square number, and so on.

NUMBER PATTERNS 2 .... page 47

1. (a) yes, yes, yes
   (b) You get a square number.
   (c) Answers may vary. Sample: ● ○ ○
   ● ● ○
   ● ● ●
   (d) $T_2 + T_3 = S_3$
   $T_3 + T_4 = S_4$, etc.
2. (a)

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

   (b) third diagonal (starting with 1s in the third row)
   (c) diagonally below last triangular number, toward the center of the triangle

MARKET DAYS .... page 49

1. 79 eggs
2. 120 at first market, 50 at second market
3. 36 chickens
4. There must be 12 people and only whole loaves.

whole loaves	half-loaves	quarter-loaves	total loaves
2	2	8	5
4	4	4	9
6	2	4	8
8	2	4	10
10	1	6	12

SOLUTIONS

Note: *Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

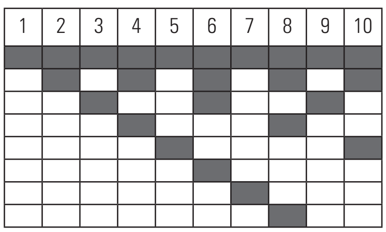
There would be 5 men (2 loaves each), 1 woman (half-loaf), and 6 children (quarter-loaf each).

THE FARMERS MARKET .... page 51

1. Use "try and adjust" by choosing a likely number and then 25 less than half of it. He sold 398 avocados on Saturday and 174 avocados on Sunday.
2. He had 447 tomatoes at the start of the market.
3. He brought 66 pumpkins to the market.

ABSTRACT ART .... page 53

1. Use counters or color the squares on a grid:



This shows the factors of the number of the column in which they occur.
Checking the factors of each number 1–50 shows that 48 has the most entries in its column.

2. columns whose numbers are prime numbers (2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47)
3. 4, 9, 25, 49 (the square numbers)
4. (a) 1
   (b) Factors occur in pairs except for the square numbers.
5. 144 (It has the largest number of factors of the numbers 1–150.)

TIME TAKEN .... page 54

1. 12 mi per hour

Hours	10 mi/hr	15 mi/hr	Difference
1	10	15	5
2	20	30	10
3	30	45	15
4	40	60	20
5	50	75	25

2. 1 hour 10 minutes later
3. Ernest got home 24 minutes earlier than usual, so he drove 12 minutes less than usual in each direction. He normally meets Anna at 6:30, so today they met up 12 minutes earlier, or at 6:18. Anna walked from 5:30 until 6:18, or 48 minutes.

CHANGING LOCKERS.................................... page 55

1. (a) Answers will vary.
   (b) 1, 4, 9
   (c) 1, 4, 9, 16, 25, 49
   (d) Square numbers have an odd number of factors, and these lockers will be open.
   (e) 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961
   (f) 2, 3, 5, 7, 11, 13
   (g) They are prime numbers and have only 2 factors.

SCHOOL RECORDS .... page 57

1. 265 pages
2. 157 times
3. page 133
4. 291 digits

THE TOWN'S HISTORY .... page 58

1. 3,001 digits
2. 303 times
3. 1 (since it occurs most in the 4 digit numbers)
4. 1,528 digits

TEAM PHOTOS .... page 59

1. There are 53 players.
2. 7 pages, 84 photos
3. Make a list of the number of pages with 5 and 2 extra and check how many pages with 8 photos this gives:

Number of pages	Number of photos	Pages with 8 photos	Photos on last page
10	47	5	7
11	52	6	4
12	57	7	1
13	62	7	6
14	67	8	3

There were 67 photos.

4. Not possible

SOLUTIONS

***Note:** Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

AFTER WORK .... page 61

1. (a) Cath
   (b) Cath 13 minutes, Jared 14 minutes
   (c) No, the fast elevator takes 19 minutes; the slow elevator takes 20 minutes.

Time	5:00	5:01	5:02	5:03	5:04	5:05	5:06	5:07	5:08	5:09	5:10
Fast	8	7	7	7	7	6	5	5	5	5	4
Slow	8		7		6	6	6	6		5	

Time	5:11	5:12	5:13	5:14	5:15	5:16	5:17	5:18	5:19	5:20
Fast	4	4	4	3	2	2	2	2	1	
Slow	4		3	3	3	3		2		1

   (d) The slow elevator takes 22 minutes; the fast elevator takes 23 minutes.

Time	5:00	5:01	5:02	5:03	5:04	5:05	5:06	5:07	5:08	5:09	5:10	5:11
Jared	9	8	8	8	8	7	6	6	6	6	5	5
Cath	9		8		7	7	7	7		6		5

Time	5:12	5:13	5:14	5:15	5:16	5:17	5:18	5:19	5:20	5:21	5:22	5:23
Jared	5	5	4	3	3	3	3	2	2	2	2	1
Cath		4	4	4	4		3		2		1	

2. 25 people

THE FISH MARKET .... page 63

1. The cost of the fish and shrimp is $64, and the cost of the shrimp and oysters is $33.

   Use a table to "try and adjust" an amount for the oysters:

	Fish	Shrimp	Oysters	Fish & Shrimp	Shrimp & Oysters	Fish & Oysters
Try	40	24	9	64	33	$49 – too little
Try	45	19	14	64	33	$59 – too little
Try	46	18	15	64	33	$61 – too much
Try	45.50	18.50	14.50	64	33	$60

   Fish $45.50; shrimp $18.50; oysters $14.50

2. no, if the market is open 7 days
   yes, if the market is open 5 or 6 days per week
3. Use a table to "try and adjust" for the price of the fish, realizing that multiples of 2 and 3 (i.e., 6) are needed. Since the salmon is more expensive, start with $60: $144 for 2 salmon and 3 cod.

	Salmon	Cod	3 Salmon & 2 Cod
Try	$60	$8	$196 – too much
Try	$54	$12	$186 – too much
Try	$48	$16	$176 – too much
Try	$42	$20	$166 – too much
Try	$36	$24	$156

   Salmon is $36 per kg; cod is $24 per kg.

4. (a) The fourth problem is similar to the first.

	Sardines	Squid	Mussels	Sardines & Squid	Sardines & Mussels	Squid & Mussels
Try	$40	$10	$3	$50	$43	$13 – too little
Try	$35	$15	$8	$50	$43	$23 – too little
Try	$30	$20	$13	$50	$43	$33 – too much
Try	$32.50	$17.50	$10.50	$50	$43	$28 – too much
Try	$33.50	$16.50	$9.50	$50	$43	$26

   Sardines are $33.50 per tray; squid is $16.50 per bag; mussels are $9.50 per box.

   (b) $4.75
   (c) $5.50

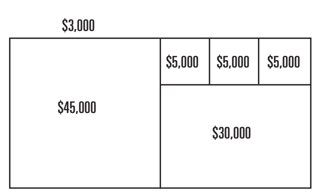
MONEY MATTERS .... page 65

1.

	Mother	Son	Mother + daughter	Son + daughter
Try	$120	$100	$80	$180 – too much
Try	$130	$90	$70	$160 – too much
Try	$140	$80	$60	$140 – too little
Try	$135	$85	$65	$150 – correct

   Mother spent $135, son spent $85, daughter spent $65

2. (a) Brian got $76, Carol got $54
   (b) Their aunt gave them $220.
3.



Daughter got $48,000; grandson got $30,000; and great-grandchildren got $5,000 each.

SOLUTIONS

Note: *Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

SCORING POINTS .... page 66

1. 8

Correct	Not Correct/ Not Answered	Score
10	0	50
9	1	42
8	2	34
7	3	26

2. 50, 42, 34, 26, 18, 10, 2

3.

Correct	Not Correct	Not Answered	Score
20	0	0	60
19	0 1	1 0	57 56
18	0 1 2	2 1 0	54 53 52
17	0 1 2	3 2 1	51 50 49
17	3	0	48
16	0	4	48
	1 2 3	3 2 1	47 46 45

17 correct and 0 unanswered, or 16 correct and 4 unanswered

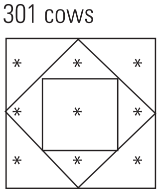
4. 3 questions were not answered correctly.

incorrect	correct	prize
1	7	$1250
2	14	$2500
3	21	$3750

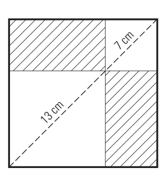
5. Adding 1st + 2nd, 1st + 3rd, 2nd + 3rd gives 2(1st + 2nd + 3rd) = 194.
   97 people attended the first 3 shows, so 21 came to the 4th show.

PUZZLE SCROLLS 1 .... page 67

1. Jill
2. 7 minutes
3. 301 cows
4. 3946
5.



6. 91 cm²



Area of whole square is half 20^2 or 200 cm^2.

Area of square with diagonal 13 is half 13^2 or 84.5 cm^2.

Area of square with diagonal 7 is half 7^2 or 24.5 cm^2.

Area of shaded parts is 200 – 84.5 – 24.5 or 91 cm^2.

TRAINING RUNS .... page 69

1. In one minute, Heather jogs 120 m, Hannah jogs 80 m, and Helen jogs 60 m. They reach the starting line after jogging 240 m.

Time in minutes	Heather	Hannah	Helen
1	120	80	60
2	start	160	120
3	120	start	180
4	start	80	start
5	120	160	60

The table can be continued until, after 12 minutes, they are all at the start together.

Looking for a pattern in the table—Heather is at the start every 2 minutes, Hannah every 3 minutes, and Helen every 4 minutes—also shows that they will be at the start together after 12 minutes.

2. (a)

Distance jogged	Len	Liam	Lachlan
1	150	60	100
2	300 – start	120	200
3	450	180	300 – start
4	600 – start	240	400
5	750	300 – start	500
6	900 – start	360	600 – start

The table can be continued until, after 30 minutes, they are all at the start together. Looking for a pattern in the table—Len is at the start every 2 minutes, Liam every 5 minutes, and Lachlan every 3 minutes—also shows that they will be at the start together after 30 minutes. Len will have jogged 15 laps, a total of 4,500 m or 4 km 500 m. Liam will have jogged 6 laps, a total of 1,800 m or 1 km 800 m. Lachlan will have jogged 10 laps, a total of 3,000 m or 3 km.

(b) 8:00 a.m.

(c) 8:30; Len 9 km, Liam 3.6 km, Lachlan 6 km

SOLUTIONS

Note: *Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

RIDING TO WORK .... page 70

1.

Time in minutes	Gary	Jeff
1	600	400
2	1,200 – start	800
3	1,800	1,200 – start
4	2,400 – start	1,600
5	3,000	2,000
6	3,600 – start	2,400 – start

After 6 minutes – Gary 3,600 m, Jeff 2,400 m

2. 8 (including the beginning)
3.

Time	24 seconds	48 seconds	72 seconds	96 seconds	120 seconds	144 seconds	168 seconds
Jeff	$\frac{3}{7}$	$\frac{6}{7}$	$\frac{9}{7}$	$\frac{12}{7}$	$\frac{15}{7}$	$\frac{18}{7}$	$\frac{21}{7}$ 3 laps
Gary	$\frac{4}{7}$	$\frac{8}{7}$	$\frac{12}{7}$	$\frac{16}{7}$	$\frac{20}{7}$	$\frac{24}{7}$	$\frac{28}{7}$ 4 laps

(a) Gary cycles around the track in 42 seconds.
(b) after 2 minutes 48 seconds
(c) Jeff 3, Gary 4

4. 15 laps
5. (a) 3 minutes 30 seconds
   (b) $17\frac{1}{7}$ laps

BIKE TRACKS .... page 71

1.

Time	40 seconds	80	120	160	200	240	280	320	360
Gina	$\frac{4}{9}$	$\frac{8}{9}$	$\frac{12}{9}$	$\frac{16}{9}$	$\frac{20}{9}$	$\frac{24}{9}$	$\frac{28}{9}$	$\frac{32}{9}$	$\frac{36}{9}$ 4 laps
Georgia	$\frac{5}{9}$	$\frac{10}{9}$	$\frac{15}{9}$	$\frac{20}{9}$	$\frac{25}{9}$	$\frac{30}{9}$	$\frac{35}{9}$	$\frac{40}{9}$	$\frac{45}{9}$ 5 laps

90 seconds

2. After 6 minutes
3. Gina 4, Georgia 5
4. 6 laps
5. 60 days later
6. 2 (including the first Saturday)
7. Saturday, Wednesday
8. Every 20 days
9. Every 12 days
10. Every 10 days

BEACH CARNIVAL .... page 73

1. There are 50 shells to collect:

Shell number	Distance run	Total distance (meters)
1	4	4
2	8	4 + 8
3	12	4 + 8 + 12
4	16	4 + 8 + 12 + 16
5	20	4 + 8 + 12 + 16 + 20
50	100	4 + 8 + 12 + 16 + 20 + ... + 100

Look for a pattern
4
4(1 + 2)
4(1 + 2 + 3)
4(1 + 2 + 3 + 4)
4(1 + 2 + 3 + 4 + 5)
4(1 + 2 + 3 + 4 + 5 +... + 50)

This can be totaled to give the answer of 5,100 m, or 5 km 100 m.

2. Jeff ran 3,612 m, or 3 km 612 m.

COLORED CUBES .... page 75

1. 9 of each color
2. (a) Answers will vary according to which color cube was at the center of the original square.
   (b) yes
3. 16
4. yes

GROWING CUBES .... page 76

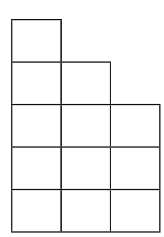
1. 7
2. 19
3. 37
4. Answers will vary.
5. 1, 7, 19, 37, 61, 91, 127 ...
6. Each number of small cubes you need to add is a multiple of 6.

1 7 19 37 61 91 127 ...

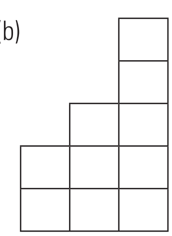
6 12 18 24 30 36

VIEWING CUBES .... page 77

1. (a)



(b)



2. Teacher check

SOLUTIONS

Note: *Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

3. (a) Total 14

2	2	1
3	2	
2	1	1

(b) Total 17

3	2	1	2	1	1
3	2	1	1		

(c) Total 11

			2	1
		2	1	
2	1	1		
1				

(d) Total 14

4			
1			1
3	1	1	3

4. Teacher check

SURFACE AREA .... page 79

1. Steps are 0.4 m^2, rises are 0.2 m^2, ends are 0.02 m^2
Area painted = 3.6 m^2
2. 6.6 m^2
3. 78 cm^2
4. Yes – 72 cm^2
5. (a) 144 cm^2
(b) Hole would be $2 \times 2 \times 2$ – surface area would be 120 cm^2
6. 1 cube removed – 174 cm^2
9 cubes removed – 222 cm^2
7. Surface 1 cube, hole 1 cube – 192 cm^2
Surface 9 cubes, hole 1 cube – 240 cm^2
Surface 9 cubes, hole 9 cubes – 168 cm^2

VOLUME AND SURFACE AREA .... page 80

1. (a) inside the prism
(b) a smaller prism $3 \times 5 \times 7$ (the prime factors of 105)
(c) Whole prism is $5 \times 7 \times 9$
315 cm^3
(d) 2 of each side: 5×7, 5×9, 7×9
286 cm^2
2. $363 = 3 \times 11 \times 11$, so whole prism is $5 \times 13 \times 13$.
Volume is 845 cm$^{3.}$
Surface area is 598 cm^2.
3. (a) $60 = 2 \times 2 \times 3 \times 5$
Internal prism could be:
$4 \times 3 \times 5$ cubes
$2 \times 6 \times 5$ cubes
$2 \times 2 \times 15$ cubes
or $2 \times 3 \times 10$ cubes

(b)

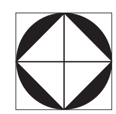
Internal Prism	Volume of Prism	Surface Area of Prism
$4 \times 3 \times 5$	$210 \times 8 = 1{,}680$ cm^3	856 cm^2
$2 \times 6 \times 5$	1,792 cm^3	928 cm^2
$2 \times 2 \times 15$	2,176 cm^3	1,216 cm^2
$2 \times 3 \times 10$	1,920 cm^3	1,024 cm^2

SURFACE AREA AND VOLUME .... page 81

1. (a) 600 cm^2
(b) 792 cm^3
(c) surface area 680 cm^2, volume 992 cm^3
(25 more blocks – equivalent to 1 more layer, extra volume is 200 cm^3, extra area is 80 cm^2)
2. (a) volume 1,566 cm^3, surface area 990 cm^2
(b) (i) volume 360 cm^3, surface area 416 cm^2
(ii) volume 1215 cm^3, surface area 900 cm^2
(iii) volume 3,584 cm^3, surface area 2,048 cm^2
(c) Teacher check

DESIGNING SHAPES..................................... page 83

1. (a) Area is 144 cm^2. The side length of the small square is half the side length of the large square.
(b) Area of each is 108 cm^2.
2. Drawing the diagonals of the square shows 8 triangles in the large square, so each has an area of 18 cm^2. The small square has an area of 4 of these triangles, or 72 cm^2.



3. (a) 272 cm^2
(b) 112 cm

DIFFERENT DESIGNS .... pages 84

1. (a) 48 in.^{2}
(b) 156 in.^{2}
(c) 75 in.
2. (a) 24 in.
(b) 36 in.^{2}
3. The perimeter of the large square is 36 in., the area is 81 in.^{2}
4. The perimeter of the large square is 64 in., the area is 256 in.^{2}
5. Perimeter of each rectangle is 72 in.
Perimeter of the square is 120 in.

THE CITY SQUARE .... pages 85

1. The 4 triangles can be arranged to form a square with sides 18 m. The area of each lawn is 81 m^2.

SOLUTIONS

Note: *Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

2. 108 cm^2
3. Lawn area is 441 m^2.

ROLLING ALONG .... page 87

1. (a) 12.5 m
   (b) 256 times
   (c) 150 m per day, so the 22nd day, a Saturday
2. (a) gear B – clockwise; gear C – counterclockwise
   (b) gear B – 6 rotations; gear A – 4 rotations
   (c) 9 times

RED ROCK ADVENTURES .... page 89

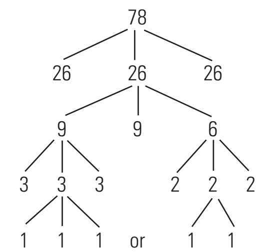
1. RC47, RC51, DS47, DS51
2. RC23, RC32, DS23, DS32
3. RC68, RC74, DS68, DS74
4. *RC47 – 10:25 a.m.*
   leave home at 10:20, arrive 11:35, plane at 12:35
   RC51 – 11:25 a.m.
   leave home at 11:35, arrive 12:50, plane at 1:50
   DS47 – 9:25 a.m.
   leave home at 9:25, arrive 10:40, plane at 11:40
   DS51 – 12:00 p.m.
   leave home at 12:00, arrive 1:15, plane at 2:15
5. Answers will vary. Discussions need to consider time needed to get to the hotel if flight or luggage is delayed.

CALENDAR CALCULATIONS .... page 91

1. They work together on Saturday of week 5.
2. Thursday.
3. a leap year
4. Wednesday. Add 366 + 365 + 365 + 365 and divide by 7. Remainder is 5, so 5 days after Friday.
5. Monday, Tuesday, Wednesday, or Thursday

BALANCING BUSINESS .... page 92

1. Each full box has 36 screws
2. 4 uses of the balance:
   Divide the 78 bags into 26, 26, and 26.



1) Weigh 26 vs. 26 with 26 left over. If equal, proceed with the 26 left over. If unequal, proceed with the heavier 26.
2) Weigh 9 vs. 9 with 6 left over. If equal, proceed with the 6 left over. If unequal, proceed with the heavier 9.
3) Weigh (if 9) 3 vs. 3 with 3 left over, or (if 6) 2 vs. 2 with 2 left over. If equal, proceed with the group left over. If unequal, proceed with the heavier group.
4) Weigh (if 3) 1 vs. 1 with 1 left over, or (if 2) 1 vs. 1. If equal, the one left over is the heavy bag. If unequal, the heavier one is his answer.

PUZZLE SCROLLS 2.. page 93

1. 168 cm
2. Sunday
3. 8:06 a.m.
4. floor 2
5. About 53 mi
6. 68.5 in.

WORLD CITIES WEATHER............................ page 95

1. Tokyo
2. 175/365 or about 47.9% (about 1 day out of 2)
3. London
4. 270/365 or about 74% (about 3 days out of 4)
5. Mumbai
6. Athens, Mumbai, Istanbul, Madrid, Tokyo
7. Copenhagen, 48.2° F
8. about 66%, or about 2 days out of 3
9. about 23%, or about 1 day out of 4
10. about 0.16 in.

SHOWTIME.. page 96

1. $\frac{10}{27}$
2. $\frac{3}{27}$ or $\frac{1}{9}$
3. $\frac{2}{7}$
4. $\frac{5}{7} \times \frac{5}{7} \times \frac{5}{7} \times \frac{5}{7}$ or $\frac{625}{2{,}401}$
5. $\frac{1}{3}$
6. 1 in 19,683
7. In terms of probability, no. Discussion should focus on the actual likelihood of customers buying 1 or more hot dogs in real life.

SOLUTIONS

Note: *Many solutions are written statements rather than simply numbers. This is to encourage teachers and students to solve problems in this way.*

PROBABLY TRUE .... page 97

1. 40 cards must be drawn.
2. Probability of drawing blue from bag 1 is $\frac{3}{5}$.
   Probability of drawing blue from bag 2 is $\frac{2}{3}$.
   Two-thirds is greater than $\frac{3}{5}$, so he should choose bag 2.
3. $\frac{15}{100}$ or $\frac{3}{20}$
4. Probability of drawing blue or yellow must be $\frac{1}{4}$.
   There are 9 blue and yellow marbles, so there must be 36 altogether. 27 must be red. Need 24 more red marbles.
5. Probability for Carly is $\frac{4}{9}$, for Toby is $\frac{3}{8}$.
   Probability of both is $\frac{12}{72}$ or $\frac{1}{6}$
6. $\frac{8}{20}$ or $\frac{2}{5}$
7. $\frac{4}{16}$ or $\frac{1}{4}$

TANK WATER .... page 99

1. 4 weeks, 4 days
2. 3 weeks
3. 4 teams would take more than 3 weeks. 5 teams would finish in 2.5 weeks, so 5 teams would be needed.

SQUARE-DEAL NURSERY .... page 100

1. One way is to try and adjust:
   68 trays

	9-Plant Trays	16-Plant Trays	Number of Plants
Try	50	18	738 – too few
Try	40	28	808 – too few
Try	30	38	878 – too few
Try	25	43	913 – too few
Try	20	48	948 – too many
Try	23	45	927

   23 3 × 3 trays
   45 4 × 4 trays

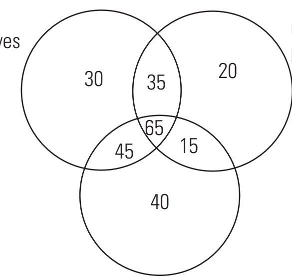
2. One ways is to try and adjust:
   300 trays

	4-Plant Trays	9-Plant Trays	Number of Plants
Try	100	200	2,200 – too many
Try	200	100	1,700 – too many
Try	250	50	1,450 – too few
Try	245	55	1,475 – too many
Try	246	54	1,470

   54 9-plant trays and 246 4-plant trays

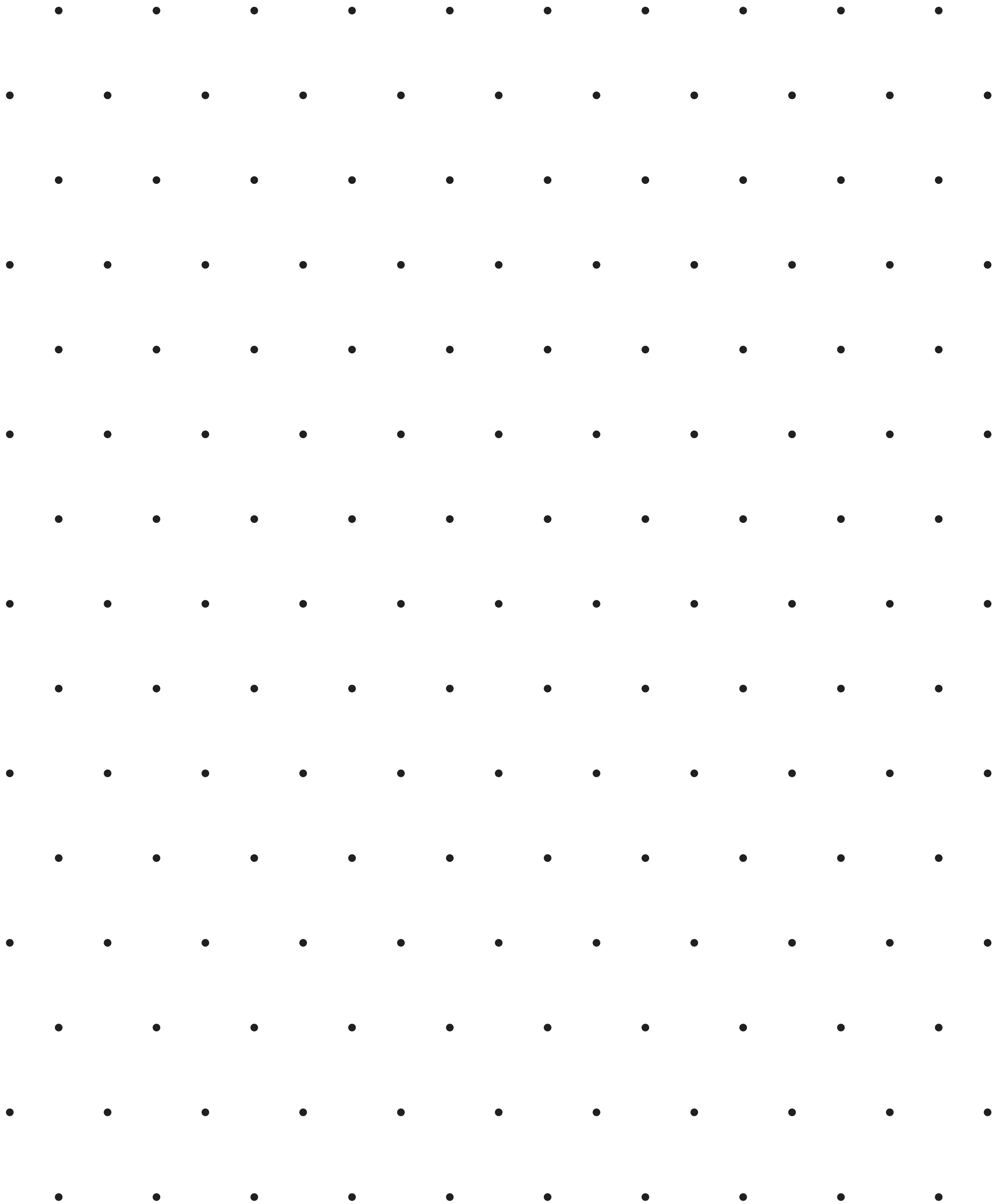
3. Refund for 1 4 × 4 tray , 2 2 × 2 trays, and 3 3 × 3 trays is $28; $896 was refunded.
   28 × 32 = 896, so 32 of this combination were recycled.
   32 4 × 4 trays, 64 2 × 2 trays, 96 3 × 3 trays
4. $350

SALAD DAYS .... page 101

1. 

2. 100
3. 30
4. 40

ISOMETRIC DOTS



0–99 BOARD

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

4-DIGIT NUMBER EXPANDER (× 5)

ones

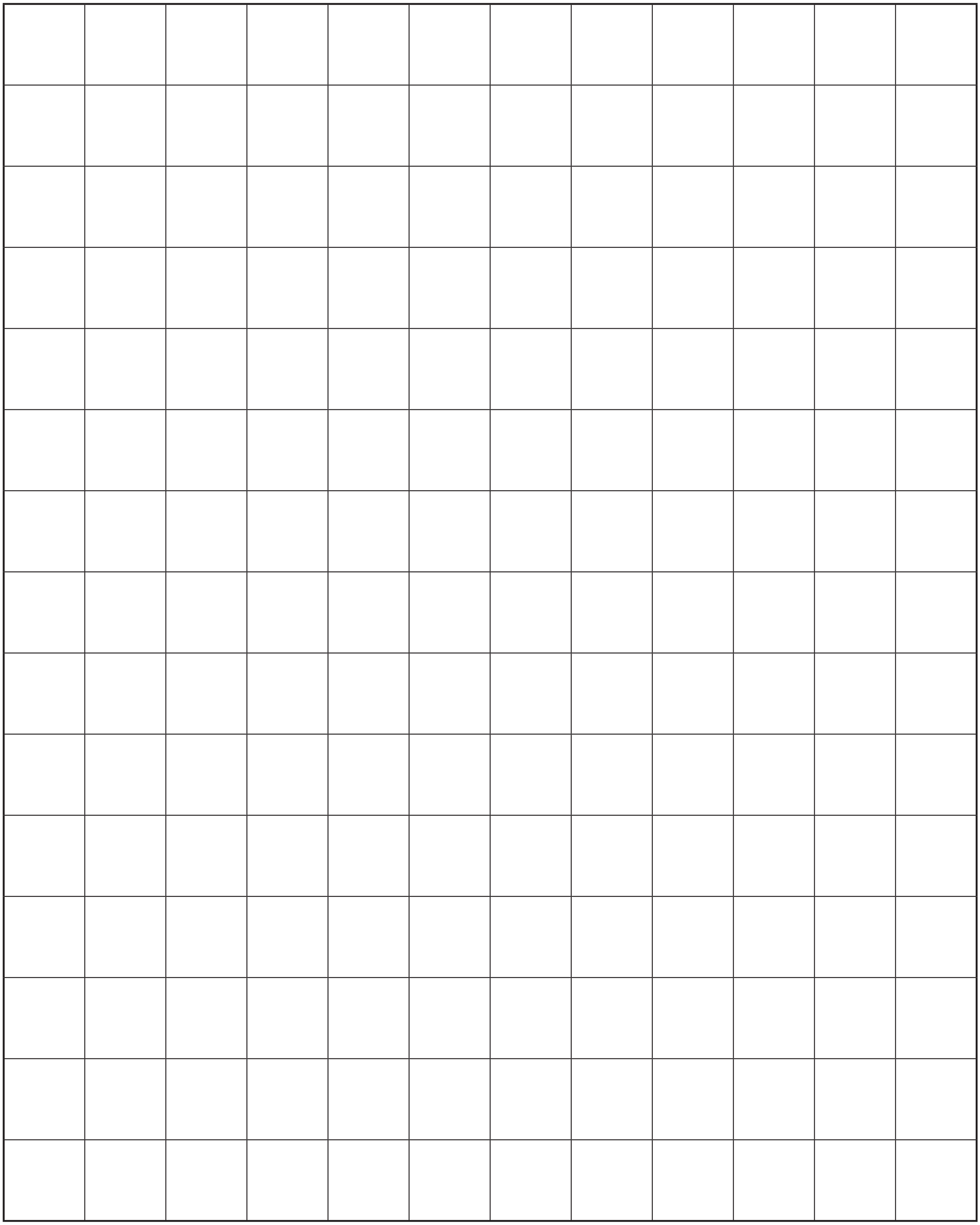
tens

hundreds

thousands

10 mm × 10 mm GRID

15 mm × 15 mm GRID



TRIANGULAR GRID